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4th International Short Course

Seismic Analysis of Structures using OpenSees: Finite Element-based Framework and Civil Engineering Applications

Methods and Formulations for Nonlinear Analysis of Reinforced Concrete Frames

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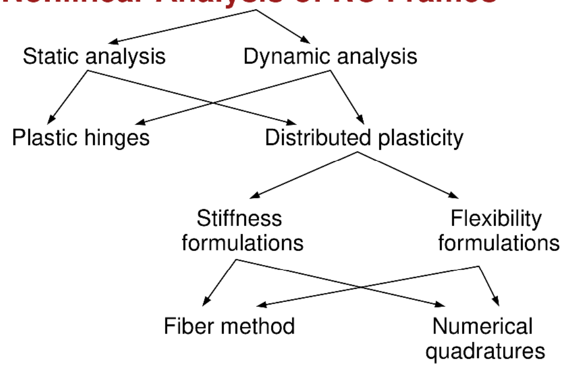
Nonlinear Analysis of RC Frames



Static analysis

The Newton method with load or displacement or arc-length controls

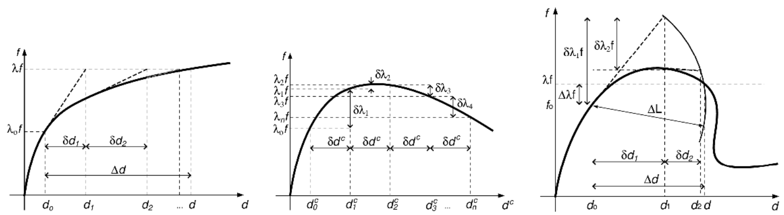
```
algorithm Newton <-initial> <-initialThenCurrent>

algorithm ModifiedNewton <-initial>
```

```
integrator LoadControl $lambda <$numIter $minLambda $maxLambda>

integrator DisplacementControl $node $dof $incr
                               <$numIter $DUmin $DUmax>

integrator ArcLength $s $alpha
```

```
test testType? <$tol> $iter <$pFlag> <$nType>
```


The Newton method with load or displacement or arc-length controls

Three numerical procedures are illustrated, based on a tangent (Newton) approach to solve the structural equilibrium.



```
algorithm Newton <-initial> <-initialThenCurrent>

algorithm ModifiedNewton <-initial>
```

-initial → (optional) use initial stiffness
-initialThenCurrent → (optional) use initial stiffness on first step, then use current stiffness for subsequent steps

The Newton method (load control)

The equilibrium equation to be solved is

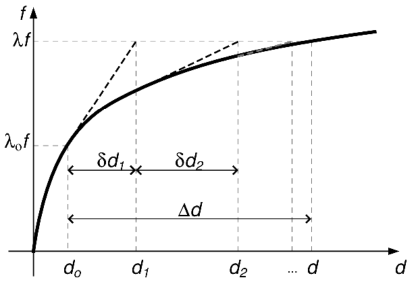
$$\mathbf{R}(\mathbf{d},\lambda) = \mathbf{f}(\mathbf{d}) - \lambda\mathbf{f} = \mathbf{0}$$

Taylor expansion of $\mathbf{R}(\mathbf{d},\lambda)$ gives

$$\mathbf{R}(\mathbf{d},\lambda) \approx \mathbf{R}(\mathbf{d}_o,\lambda_o) + \left.\frac{\partial\mathbf{R}}{\partial\mathbf{d}}\right|_{\mathbf{d}_o}(\delta\mathbf{d}) - \left.\frac{\partial\mathbf{R}}{\partial\lambda}\right|_{\lambda_o}\delta\lambda$$

where $\partial\mathbf{R}/\partial\mathbf{d} = \partial\mathbf{f}/\partial\mathbf{d} = \mathbf{K}$ and $\partial\mathbf{R}/\partial\lambda = \mathbf{f}$

$$\mathbf{R}(\mathbf{d},\lambda) \approx \mathbf{R}(\mathbf{d}_o,\lambda_o) + \mathbf{K}_o\delta\mathbf{d} - \delta\lambda\mathbf{f}$$



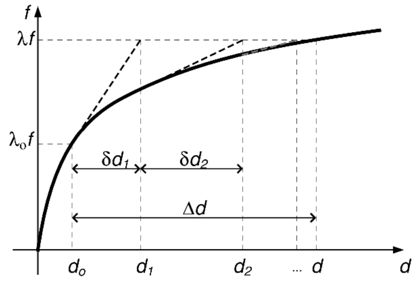
Substituting $\mathbf{R}(\mathbf{d}_o,\lambda_o) = \mathbf{f}(\mathbf{d}_o) - \lambda_o\mathbf{f}$ and $\delta\lambda = \Delta\lambda = \lambda - \lambda_o$ one has

$$\mathbf{R}(\mathbf{d},\lambda) \approx \mathbf{f}(\mathbf{d}_o) - \lambda_o\mathbf{f} + \mathbf{K}_o\delta\mathbf{d} - (\lambda-\lambda_o)\mathbf{f} = \mathbf{f}(\mathbf{d}_o) - \lambda\mathbf{f} + \mathbf{K}_o\delta\mathbf{d}$$

Equating to $\mathbf{0}$ the previous quantity and solving for $\delta\mathbf{d}$ one obtains the iterative estimate of the displacement increment

$$\delta\mathbf{d} = \mathbf{K}_o^{-1}[\lambda\mathbf{f} - \mathbf{f}(\mathbf{d}_o)] \quad \rightarrow \quad \delta\mathbf{d}_i = \mathbf{K}_{i-1}^{-1}[\lambda\mathbf{f} - \mathbf{f}(\mathbf{d}_{i-1})]$$

The Newton method (load control)



$$\delta\mathbf{d}_i = \mathbf{K}_{i-1}^{-1}[\lambda\mathbf{f} - \mathbf{f}(\mathbf{d}_{i-1})]$$

```
integrator LoadControl $lambda <$numIter $minLambda $maxLambda>
```

$lambda → load factor increment λ
$numIter → (optional, default = 1.0) the number of iterations to occur in the solution algorithm
$minLambda → (optional, default = λ) the min step size
$maxLambda → (optional, default = λ) the max step size

The Newton method (displacement control)

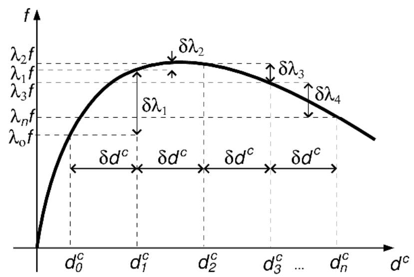
The equilibrium equation to be solved is

$$\mathbf{R}(\mathbf{d},\lambda) = \mathbf{f}(\mathbf{d}) - \lambda\mathbf{f} = \mathbf{0}$$

whose Taylor expansion gives again

$$\mathbf{R}(\mathbf{d},\lambda) \approx \mathbf{f}(\mathbf{d}_o) - \lambda_o\mathbf{f} - \delta\lambda\mathbf{f} + \mathbf{K}_o\delta\mathbf{d}$$

where the load increment factor $\delta\lambda$ has been factored out.



In this case displacements are split into an unknown part $\mathbf{d}_o^u$ and one controlled displacement d_o^c so that the equation $\mathbf{R}(\mathbf{d},\lambda) = \mathbf{0}$ becomes

$$\mathbf{f}(\mathbf{d}_o^u,\, d_o^c) - \lambda_o\mathbf{f} - \delta\lambda\mathbf{f} + [\mathbf{K}_o^u|\mathbf{K}_o^c]\begin{bmatrix}\delta\mathbf{d}^u\\ \delta d^c\end{bmatrix} = \mathbf{0}$$

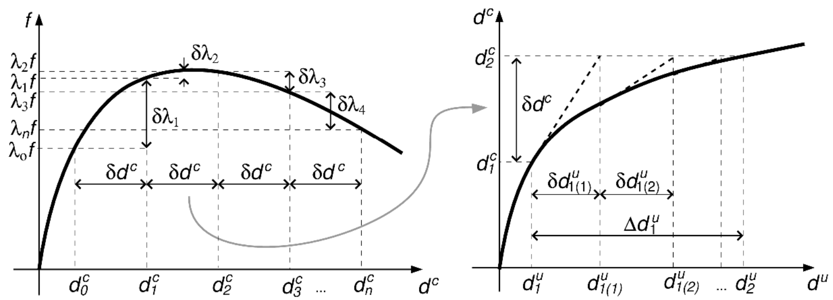
This system of equations is rewritten by factoring $\delta\mathbf{d}^u$ and $\delta\lambda$

$$\mathbf{f}(\mathbf{d}_o^u,\, d_o^c) - \lambda_o\mathbf{f} + [\mathbf{K}_o^u\,|-\mathbf{f}]\begin{bmatrix}\delta\mathbf{d}^u\\ \delta\lambda\end{bmatrix} + \mathbf{K}_o^c\delta d^c = \mathbf{0}$$

and is solved as

$$\begin{bmatrix}\delta\mathbf{d}^u\\ \delta\lambda\end{bmatrix} = [\mathbf{K}_o^u\,|-\mathbf{f}]^{-1}[\lambda_o\mathbf{f} - \mathbf{f}(\mathbf{d}_o^u,\, d_o^c) - \mathbf{K}_o^c\delta d^c]$$

The Newton method (displacement control)



$$\begin{bmatrix}\delta\mathbf{d}^u_{j(i)}\\ \delta\lambda_{j(i)}\end{bmatrix} = [\mathbf{K}^u_{j(i-1)}\,|-\mathbf{f}]^{-1}[\lambda_{j(i-1)}\mathbf{f} - \mathbf{f}(\mathbf{d}^u_{j(i-1)},\, d^c_{j(i-1)}) - \mathbf{K}^c_{j(i-1)}\Delta d^c]$$

```
integrator DisplacementControl $node $dof $incr
                         <$numIter $DUmin $DUmax>
```

$node → node to control
$dof → degree of freedom at the node
$incr → first displacement increment δd
$numIter → (optional, default = 1.0) the number of iterations to occur in the solution algorithm
$DUmin → (optional, default = δd) the min step size
$DUmax → (optional, default = δd) the max step size

The Newton method (arc-length control)

The equilibrium equation to be solved is

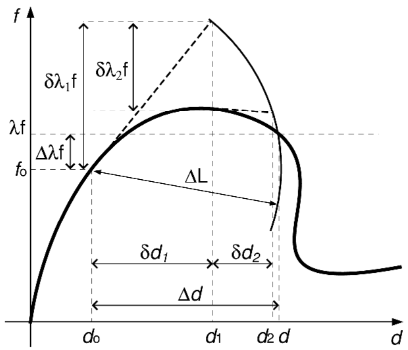
$$\mathbf{R}(\mathbf{d},\lambda) = \mathbf{f}(\mathbf{d}) - \lambda\mathbf{f} = \mathbf{0}$$

whose Taylor expansion gives again

$$\mathbf{R}(\mathbf{d},\lambda) \approx \mathbf{R}(\mathbf{d}_o,\lambda_o) + \mathbf{K}_o\delta\mathbf{d} - \delta\lambda\mathbf{f}$$

In this case the increment of the load multiplier $\delta\lambda$ is not assigned *a priori*, but it represents a further unknown added to the set of nodal displacements $\mathbf{d}$.

Hence, a further equation is introduced, expressing the distance, in the $\mathbf{d} - \mathbf{f}$ space, between the initial point $(\mathbf{d}_o, \mathbf{f}_o)$ and $(\mathbf{d}, \lambda\mathbf{f})$, expressed as:



$$a(\mathbf{d},\lambda) = \Delta\mathbf{d}\cdot\Delta\mathbf{d} + \alpha^2\Delta\lambda^2\mathbf{f}\cdot\mathbf{f} - \Delta L^2 = 0$$

where α is a scaling parameter transforming the force units to displacement units.

Taylor expansion of $a(\mathbf{d},\lambda)$ gives

$$a(\mathbf{d},\lambda) \approx a(\mathbf{d}_o,\lambda_o) + \left.\frac{\partial a}{\partial \mathbf{d}}\right|_{\mathbf{d}_o}\delta\mathbf{d} + \left.\frac{\partial a}{\partial \lambda}\right|_{\lambda_o}\delta\lambda = a(\mathbf{d}_o,\lambda_o) + 2\Delta\mathbf{d}_o\delta\mathbf{d} + 2\alpha^2\Delta\lambda_o\mathbf{f}\cdot\mathbf{f}\delta\lambda$$

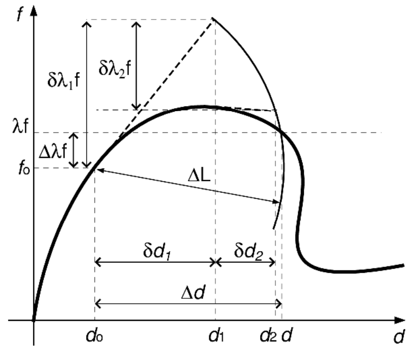
The Arc-length method

Employing the Taylor expansion of $\mathbf{R}(\mathbf{d},\lambda)$ and $a(\mathbf{d},\lambda)$ in place of the actual expression of these functions, the equilibrium equations and the arc-length form the linear system of equations

$$\begin{cases} \mathbf{R}(\mathbf{d}_o,\lambda_o) + \mathbf{K}_o\delta\mathbf{d} - \delta\lambda\mathbf{f} = \mathbf{0} \\ a(\mathbf{d}_o,\lambda_o) + 2\Delta\mathbf{d}\delta\mathbf{d} + 2\alpha^2\Delta\lambda\mathbf{f}\cdot\mathbf{f}\delta\lambda - \Delta L^2 = 0 \end{cases}$$

and is solved as

$$\begin{pmatrix}\delta\mathbf{d}\\ \delta\lambda\end{pmatrix} = -\begin{bmatrix}\mathbf{K}_o & -\mathbf{f}\\ 2\Delta\mathbf{d}^T & 2\alpha^2\Delta\lambda\mathbf{f}\cdot\mathbf{f}\end{bmatrix}^{-1}\begin{pmatrix}\mathbf{R}(\mathbf{d}_o,\lambda_o)\\ a(\mathbf{d}_o,\lambda_o)\end{pmatrix}$$

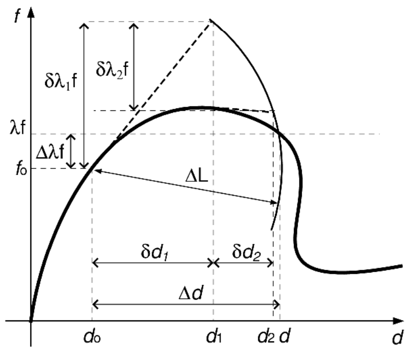


where the matrix within square brackets remains non singular even when $\mathbf{K}_o$ is singular.

Actually, we recall that the classical Newton method employs the iteration formula

$$\delta\mathbf{d} = \mathbf{K}_o^{-1}[\lambda\mathbf{f} - \mathbf{f}(\mathbf{d}_o)] = \mathbf{K}_o^{-1}\mathbf{R}(\mathbf{d}_o,\lambda_o)$$

The Newton method (arc-length control)



$$\begin{pmatrix}\delta\mathbf{d}_i\\ \delta\lambda_i\end{pmatrix} = -\begin{bmatrix}\mathbf{K}_{i-1} & -\mathbf{f}\\ 2\Delta\mathbf{d}_{i-1}^T & 2\alpha^2\Delta\lambda_{i-1}\mathbf{f}\cdot\mathbf{f}\end{bmatrix}^{-1}\begin{pmatrix}\lambda_{i-1}\mathbf{f} - \mathbf{f}(\mathbf{d}_{i-1}^u,\, d_{i-1}^c)\\ \Delta\mathbf{d}_{i-1}\cdot\Delta\mathbf{d}_{i-1} + \alpha^2\Delta\lambda_{i-1}^2\mathbf{f}\cdot\mathbf{f} - \Delta L^2\end{pmatrix}$$

```
integrator ArcLength $s $alpha
```

$s → arcLength ΔL
$alpha → scaling factor on loads

Convergence test

```
test testType? $tol $iter <$pFlag> <$nType>
```

$tol → tolerance
$iter → max number of iterations before returning failure
$pFlag → (optional, default = 0) print flag (see Command Manual)
$nType → (optional, default = 2) type of norm

Options for testType?

NormUnbalance → $|\Delta\mathbf{f}_i| < tol$

RelativeNormUnbalance → $\dfrac{|\Delta\mathbf{f}_i|}{|\Delta\mathbf{f}_o|} < tol$

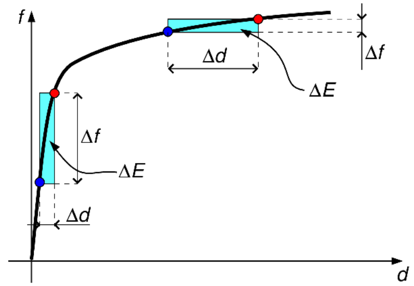
NormDispIncr → $|\Delta\mathbf{d}_i| < tol$

RelativeNormDispIncr → $\dfrac{|\Delta\mathbf{d}_i|}{|\Delta\mathbf{d}_o|} < tol$

RelativeTotalNormDispIncr → $\dfrac{|\Delta\mathbf{d}_i|}{\sum_{k=o}^{i}|\Delta\mathbf{d}_k|} < tol$

EnergyIncr → $\Delta E_i = \Delta\mathbf{d}_i\cdot\Delta\mathbf{f}_i < tol$

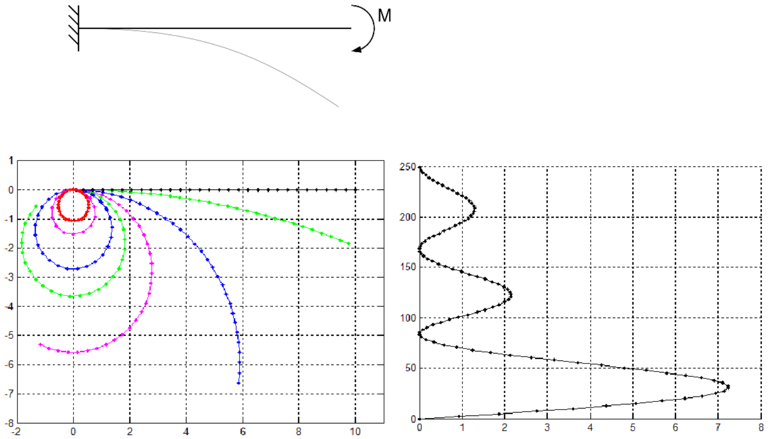
RelativeEnergyIncr → $\dfrac{\Delta E_i}{\Delta E_o} = \dfrac{\Delta\mathbf{d}_i\cdot\Delta\mathbf{f}_i}{\Delta\mathbf{d}_o\cdot\Delta\mathbf{f}_o} < tol$



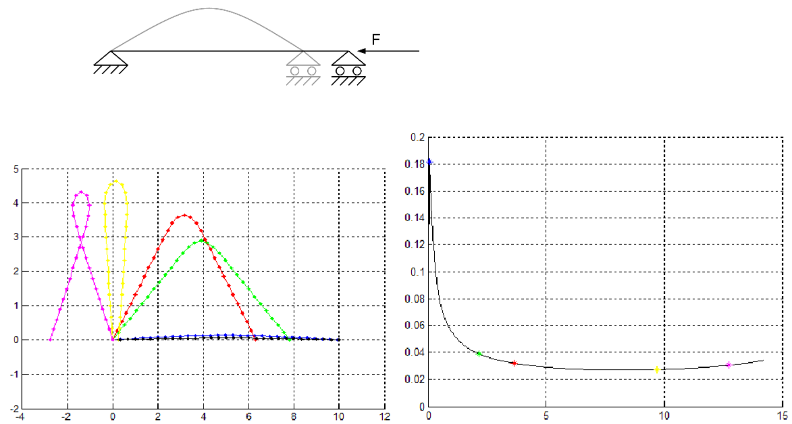
NO convergence test

```
test FixedNumIter $iter <$pFlag> <$nType>
```

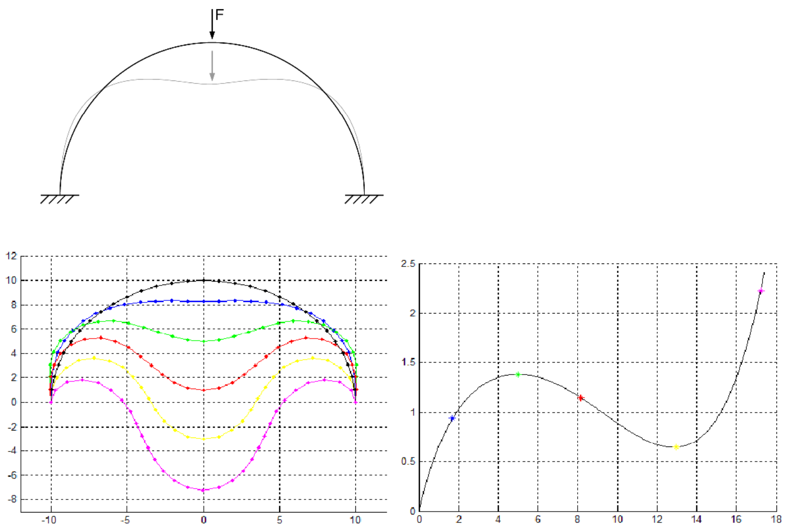
Example 1: Elastic cantilever beam in large displacements



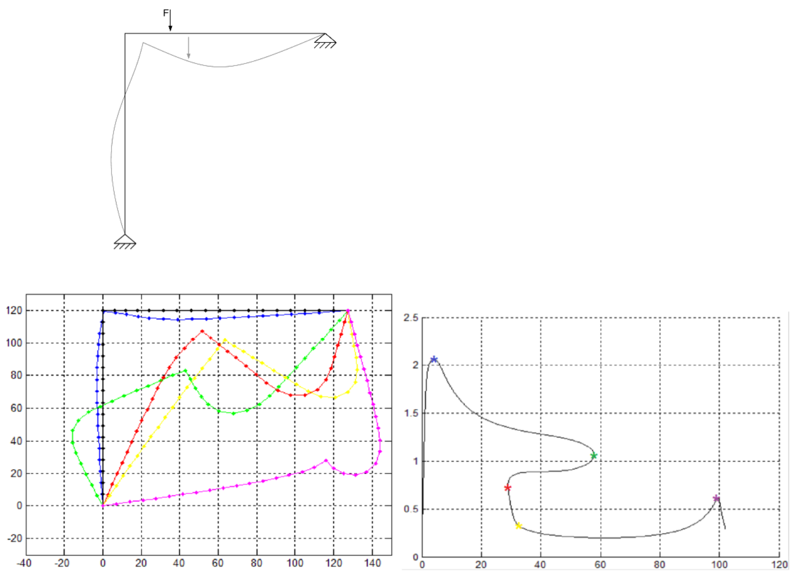
Example 2: Elastic-plastic buckling



Example 3: Snapping through arc



Example 4: 2D frame

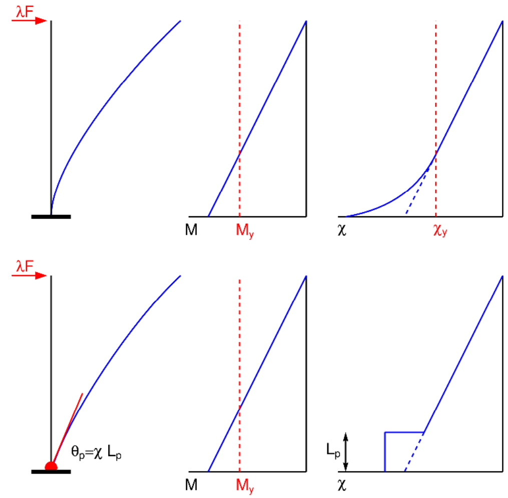


Element formulations

Plastic hinges and distributed plasticity

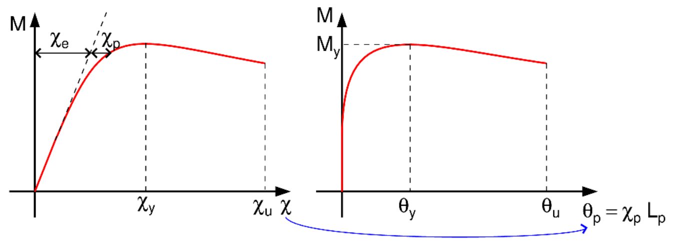
Plastic hinges: the main idea

Lamped plasticity formulations are based on the observation that many elements of a framed structure exhibit damage at their extremities.



Plastic hinges length

The plastic hinge length L_p is a fundamental parameter for defining the hinge behaviour.
It is used to transform the bending-curvature behaviour of the beam's sections into a bending-plastic rotation law of the hinges.



$$L_p = 0.1 L_v + 0.17 h + 0.24 \frac{d_f f_y}{\sqrt{f_c}} \qquad L_v = \frac{M}{T}$$

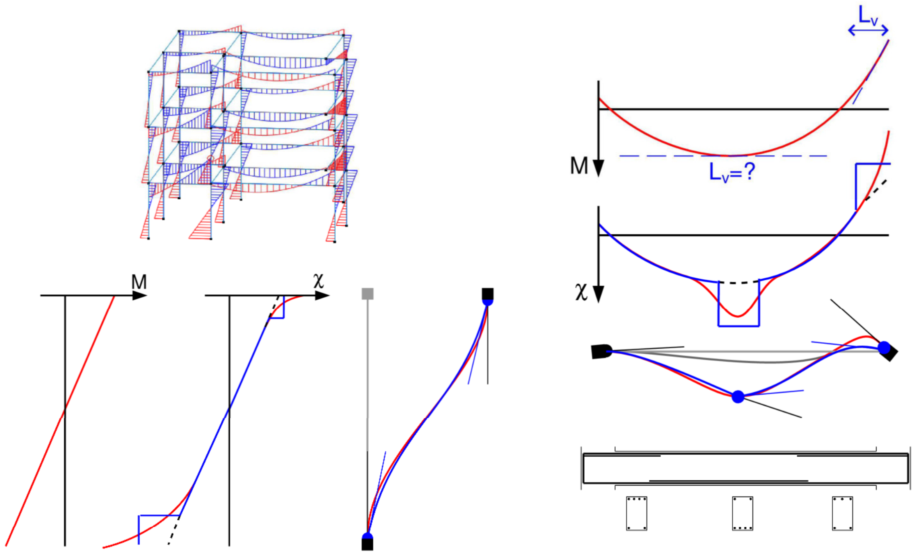
where:

$$\theta_y = \chi_y \frac{L_v}{3} + 0.0013 \left(1 + 1.5 \frac{h}{L_v}\right) + 0.13 \chi_y \frac{d_f f_y}{\sqrt{f_c}} \rightarrow \text{elastic limit rotation}$$

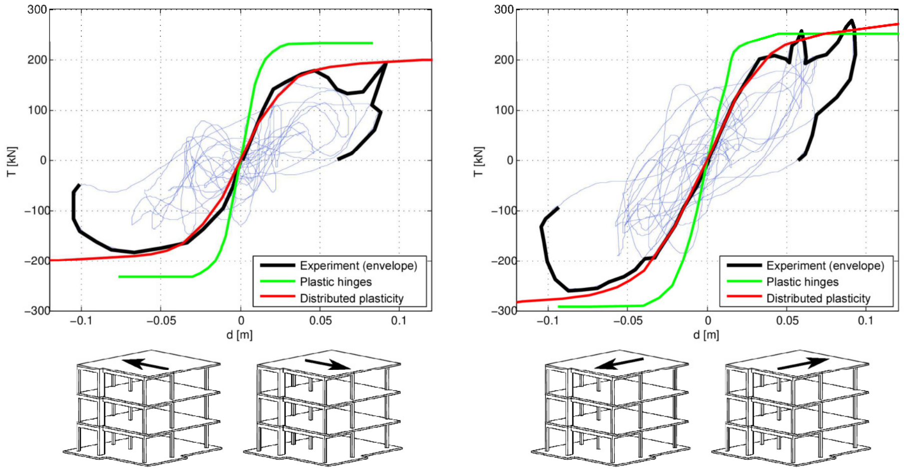
$$\theta_u = \frac{1}{1.5} \left[\theta_y + (\chi_u - \chi_y) L_p \left(1 - 0.5 \frac{L_p}{L_v}\right)\right] \rightarrow \text{ultimate rotation}$$

Considerations on the plastic hinge position

Plastic hinges are expected to open at element extremities. This is generally true only for columns.



Pushover analysis of a 3D RC frame: The SPEAR building



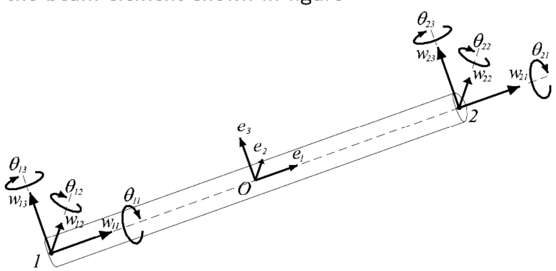
Distributed plasticity formulations

Stiffness formulations

```
element dispBeamColumn $eleTag $iNode $jNode $numIntgrPts
                       $secTag $transfTag <-mass $massDens>
                      <-cMass> <-integration $intType>
```


Distributed plasticity formulations: Stiffness formulation

Let us consider the beam element shown in figure



The differential equations governing the beam problem allow for describing displacements along the beam axis as a function of the extremity displacements by means of the element shape functions

$$w_1(x_1) = N_1(x_1)w_{11} + N_2(x_1)w_{21} \qquad \theta_1(x_1) = N_1(x_1)\theta_{11} + N_2(x_1)\theta_{21}$$

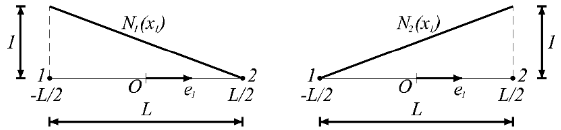
$$w_2(x_1) = H_{11}(x_1)w_{12} + H_{12}(x_1)\theta_{13} + H_{21}(x_1)w_{22} + H_{22}(x_1)\theta_{23} \qquad \theta_3(x_1) = w_2'(x_1)$$

$$w_3(x_1) = H_{11}(x_1)w_{13} - H_{12}(x_1)\theta_{12} + H_{21}(x_1)w_{23} - H_{22}(x_1)\theta_{22} \qquad \theta_2(x_1) = -w_3'(x_1)$$

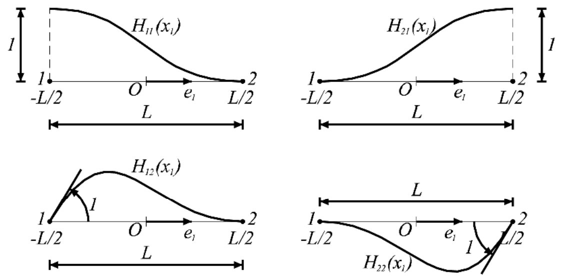
The shape functions

The shape functions are evaluated as the exact solution of the Bernoulli beam equations.

- Axial or torsional equations $K_A w_1''(x) = 0$ and $K_T \theta_1''(x) = 0 \rightarrow N''(x) = 0$



- Flexural equation $K_{F2} w_2''''(x) = 0$ and $K_{F3} w_3''''(x) = 0 \rightarrow H''''(x) = 0$



Matrix representation of displacement interpolation

Recalling that

$$w_1(x_1) = N_1(x_1)w_{11} + N_2(x_1)w_{21} \qquad \theta_1(x_1) = N_1(x_1)\theta_{11} + N_2(x_1)\theta_{21}$$

$$w_2(x_1) = H_{11}(x_1)w_{12} + H_{12}(x_1)\theta_{13} + H_{21}(x_1)w_{22} + H_{22}(x_1)\theta_{23} \qquad \theta_3(x_1) = w_2'(x_1)$$

$$w_3(x_1) = H_{11}(x_1)w_{13} - H_{12}(x_1)\theta_{12} + H_{21}(x_1)w_{23} - H_{22}(x_1)\theta_{22} \qquad \theta_2(x_1) = -w_3'(x_1)$$

we can write $\mathbf{d}^s = \mathbf{S}\mathbf{d}^e$. In components

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \left[\begin{array}{cccccc|cccccc} N_1 & 0 & 0 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & H_{11} & 0 & 0 & 0 & H_{12} & 0 & H_{21} & 0 & 0 & 0 & H_{22} \\ 0 & 0 & H_{11} & -H_{12} & 0 & 0 & 0 & H_{21} & -H_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & 0 & 0 & 0 & 0 & 0 & N_2 & 0 & 0 \\ 0 & 0 & -H_{11}' & H_{12}' & 0 & 0 & 0 & -H_{21}' & H_{22}' & 0 & 0 & 0 \\ 0 & -H_{11}' & 0 & 0 & 0 & -H_{12}' & 0 & -H_{21}' & 0 & 0 & 0 & -H_{22}' \end{array}\right] \begin{pmatrix} w_{11} \\ w_{12} \\ w_{13} \\ \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ w_{21} \\ w_{22} \\ w_{23} \\ \theta_{21} \\ \theta_{22} \\ \theta_{23} \end{pmatrix}$$

Matrix representation of strain interpolation

Similarly, strain components can be evaluated by differentiating the displacement field

$$\varepsilon_o(x) = w_1'(x) = N_1'(x_1)w_{11} + N_2'(x_1)w_{21} \qquad \theta_T'(x_1) = N_1'(x_1)\theta_{11} + N_2'(x_1)\theta_{21}$$

$$g_2(x_1) = -w_2''(x) = -H_{11}''(x_1)w_{12} + -H_{12}''(x_1)\theta_{13} + -H_{21}''(x_1)w_{22} + -H_{22}''(x_1)\theta_{23}$$

$$g_3(x_1) = -w_3''(x) = -H_{11}(x_1)w_{13} + H_{12}(x_1)\theta_{12} - H_{21}(x_1)w_{23} + H_{22}(x_1)\theta_{22}$$

we can write $\mathbf{u} = \mathbf{B}\mathbf{d}^e$. In components

$$\begin{pmatrix} \varepsilon_o \\ g_2 \\ g_3 \\ \theta_T' \end{pmatrix} = \left[\begin{array}{cccccc|cccccc} N_1' & 0 & 0 & 0 & 0 & 0 & N_2' & 0 & 0 & 0 & 0 & 0 \\ 0 & -H_{11}'' & 0 & 0 & 0 & -H_{12}'' & 0 & -H_{21}'' & 0 & 0 & 0 & -H_{22}'' \\ 0 & 0 & -H_{11}'' & H_{12}'' & 0 & 0 & 0 & -H_{21}'' & H_{22}'' & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1' & 0 & 0 & 0 & 0 & 0 & N_2' & 0 & 0 \end{array}\right] \begin{pmatrix} w_{11} \\ w_{12} \\ w_{13} \\ \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ w_{21} \\ w_{22} \\ w_{23} \\ \theta_{21} \\ \theta_{22} \\ \theta_{23} \end{pmatrix}$$

Internal forces

Internal forces can be computed as a function of the strain parameters by means of the section stiffness matrix.

In linear elasticity this relation yields $\boldsymbol{\sigma} = \mathbf{K}_E^s\mathbf{u}$

$$\begin{pmatrix} N \\ -M_3 \\ M_2 \\ M_T \end{pmatrix} = \begin{bmatrix} EA & Es_2 & Es_3 & 0 \\ Es_2 & EJ_{22} & EJ_{23} & 0 \\ Es_3 & EJ_{32} & EJ_{33} & 0 \\ 0 & 0 & 0 & GC_T \end{bmatrix} \begin{pmatrix} \varepsilon_o \\ g_2 \\ g_3 \\ \theta_T' \end{pmatrix}$$

For nonlinear material behavior, displacements along the beam axis are evaluated as a function of the extremity displacements by means of the same shape functions employed for the elastic beam.

Only the section stiffness matrix is modified by modelling the nonlinear material behaviour in terms of stress and strain increments as $\Delta\boldsymbol{\sigma} = \mathbf{K}^s\Delta\mathbf{u}$

$$\begin{pmatrix} \Delta N \\ -\Delta M_3 \\ \Delta M_2 \\ \Delta M_T \end{pmatrix} = \begin{bmatrix} A_E^T & s_{E2}^T & s_{E3}^T & 0 \\ s_{E2}^T & J_{E22}^T & J_{E23}^T & 0 \\ s_{E3}^T & J_{E32}^T & J_{E33}^T & 0 \\ 0 & 0 & 0 & K_T \end{bmatrix} \begin{pmatrix} \Delta\varepsilon_o \\ \Delta g_2 \\ \Delta g_3 \\ \Delta\theta_T' \end{pmatrix}$$

where $\mathbf{K}^s$ is the section tangent stiffness.

What does the algorithm ask to elements?

Newton method with force control

$$\Delta\mathbf{d}_i = \mathbf{K}_{i-1}^{-1}[\lambda\mathbf{f} - \mathbf{f}(\mathbf{d}_{i-1})]$$

Newton method with displacement control

$$\begin{bmatrix} \Delta\mathbf{d}_{j(i)}^u \\ \Delta\lambda_{j(i)} \end{bmatrix} = [\mathbf{K}_{j(i-1)}^u \,|\, -\mathbf{f}]^{-1}[\lambda_{j(i-1)}\mathbf{f} - \mathbf{f}(\mathbf{d}_{j(i-1)}^u, d_{j(i-1)}^c) - \mathbf{K}_{j(i-1)}^c\Delta d^c]$$

Newton method with arc-length control

$$\begin{pmatrix} \Delta\mathbf{d}_i \\ \Delta\lambda_i \end{pmatrix} = -\begin{bmatrix} \mathbf{K}_{i-1} & -\mathbf{f} \\ 2\Delta\mathbf{d}_{i-1}^T & 2\alpha^2\Delta\lambda_{i-1}\mathbf{f}\cdot\mathbf{f} \end{bmatrix}^{-1} \begin{pmatrix} \lambda_{i-1}\mathbf{f} - \mathbf{f}(\mathbf{d}_{i-1}^u, d_{i-1}^c) \\ \Delta\mathbf{d}_{i-1}\cdot\Delta\mathbf{d}_{i-1} + \alpha^2\Delta\lambda_{i-1}^2\mathbf{f}\cdot\mathbf{f} - \Delta L^2 \end{pmatrix}$$

where

$$\mathbf{f}(\mathbf{d}) = \mathop{\mathbf{A}}_{e} \mathbf{f}^e(\mathbf{d}^e) \qquad \mathbf{K} = \mathbf{K}(\mathbf{d}) = \mathop{\mathbf{A}}_{e} \mathbf{K}^e(\mathbf{d}^e)$$

Nodal forces

Element forces can be evaluated as a function of the internal forces by employing the principle of virtual works.

In particular, shape function interpolation is used within the element internal work

$$IVW = \int_{-L/2}^{L/2} \Delta\boldsymbol{\sigma}(x_1) \cdot \Delta\tilde{\mathbf{u}}(x_1) dx_1 = \int_{-L/2}^{L/2} \Delta\boldsymbol{\sigma}(x_1) \cdot [\mathbf{B}(x_1)\Delta\tilde{\mathbf{d}}^e] dx_1$$

Rearranging the last integral

$$IVW = \int_{-L/2}^{L/2} \Delta\boldsymbol{\sigma}(x_1) \cdot [\mathbf{B}(x_1)\Delta\tilde{\mathbf{d}}^e] dx_1 = \int_{-L/2}^{L/2} \mathbf{B}^T(x_1)\Delta\boldsymbol{\sigma}(x_1) dx_1 \cdot \Delta\tilde{\mathbf{d}}^e = \Delta\mathbf{f}^e \cdot \Delta\tilde{\mathbf{d}}^e = EVW$$

the element force increments can be determined

$$\Delta\mathbf{f}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \Delta\boldsymbol{\sigma} dx_1 \quad \Rightarrow \quad \mathbf{f}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \boldsymbol{\sigma} dx_1$$

where dependence of $\mathbf{B}$ and $\boldsymbol{\sigma}$ upon x_1 has been omitted for simplicity.

Element stiffness matrix

The principle of virtual work is also used to define the element stiffness matrix.

From the previous equation we have

$$IVW = \int_{-L/2}^{L/2} \Delta\boldsymbol{\sigma}(x_1) \cdot \Delta\tilde{\mathbf{u}}(x_1) dx_1 = \int_{-L/2}^{L/2} \mathbf{B}^T(x_1)\Delta\boldsymbol{\sigma}(x_1) dx_1 \cdot \Delta\tilde{\mathbf{d}}^e$$

where we can substitute $\Delta\boldsymbol{\sigma} = \mathbf{K}^s\Delta\mathbf{u}$ to have

$$IVW = \int_{-L/2}^{L/2} \mathbf{B}^T(x_1)\mathbf{K}^s(x_1)\Delta\mathbf{u}(x_1) dx_1 \cdot \Delta\tilde{\mathbf{d}}^e = \int_{-L/2}^{L/2} \mathbf{B}^T(x_1)\mathbf{K}^s(x_1)\mathbf{B}(x_1)\Delta\mathbf{d}^e \, dx_1 \cdot \Delta\tilde{\mathbf{d}}^e$$

in which the shape function interpolation of the element displacements has been used once more.

Rearranging the last term and comparing with the external virtual work

$$IVW = \int_{-L/2}^{L/2} \mathbf{B}^T(x_1)\mathbf{K}^s(x_1)\mathbf{B}(x_1) dx_1 \Delta\mathbf{d}^e \cdot \Delta\tilde{\mathbf{d}}^e = \Delta\mathbf{f}^e \cdot \Delta\tilde{\mathbf{d}}^e = EVW$$

and eliminating the virtual displacements $\Delta\tilde{\mathbf{d}}$, we obtain the incremental equilibrium equation of the element

$$\mathbf{K}^e\Delta\mathbf{d}^e = \Delta\mathbf{f}^e \quad \text{where} \quad \mathbf{K}^e = \int_{-L/2}^{L/2} \mathbf{B}^T\mathbf{K}^s\mathbf{B} \, dx_1$$

The element state determination

The solving procedure of the entire structure calls each element for evaluating the element stiffness matrix and the element forces as a function of the assigned nodal displacements.

Assigned displacements are interpolated by means of the strain operator $\mathbf{B}$

$$\mathbf{u} = \mathbf{B}\mathbf{d}^e$$

which are then used to evaluate the resultant forces and the stiffness matrix of the section

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}) \qquad \mathbf{K}^s = \mathbf{K}^s(\mathbf{u})$$

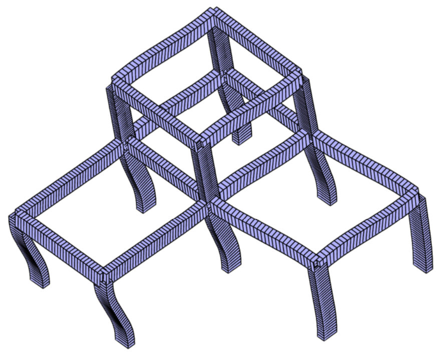
as integrals extended to the domain occupied by the element cross section.

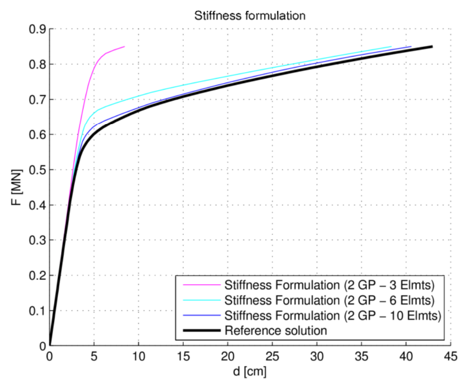
These quantities are then used to evaluate the element forces and the element stiffness matrix

$$\mathbf{f}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \boldsymbol{\sigma} \, dx_1 \qquad \mathbf{K}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \mathbf{K}_T \mathbf{B} \, dx_1$$

which are assembled to form the entire vector of structural forces and stiffness matrix needed to perform the structural analysis.

Pushover analysis of a 3D RC frame



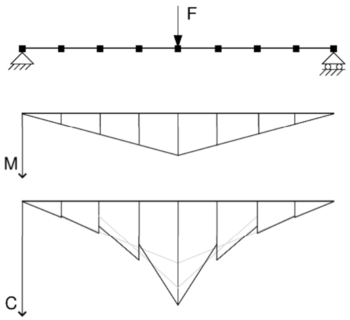


Stiffness formulations tend to overestimate structural stiffness and strength.
The result is more accurate if more elements are used to model each member.

Stiffness vs. flexibility formulations

Stiffness formulations

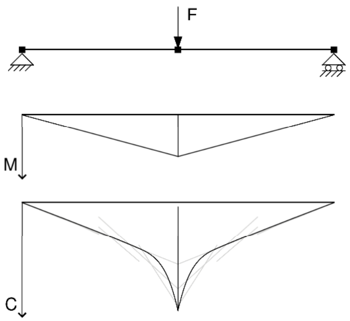
Strains are determined by interpolating nodal displacements.



Cubic shape functions → Linear curvature on each element.
Bending moment can be evaluated *a posteriori* by element equilibrium.

Flexibility formulations

Stresses are determined by interpolating nodal forces.



Linear distribution of bending moment is exact in absence of distributed loads.
Nonlinear curvature is determined from the bending moments according to the constitutive relationship.

Distributed plasticity formulations

Flexibility formulations

```
element forceBeamColumn $eleTag $iNode $jNode $numIntgrPts
                        $secTag $transfTag <-mass $massDens>
                       <-iter $maxIters $tol> <-integration $intType>
```

Why the additional (optional) command -iter $maxIters $tol?
[Default values are $maxIters=10 and $tol=10E-12]

Elimination of rigid body modes

The global structural problem is more conveniently defined in terms of nodal displacements and solved by means of a procedure based on the evaluation of the tangent stiffness and force residuals of the structural model.

The stiffness matrix of each element is evaluated as the inverse of its flexibility matrix.

$$\mathbf{K}^e = [\mathbf{F}^e]^{-1}$$

hence, avoid matrix singularity, the formulation needs to be derived in a reference free of rigid body modes.

This is done by employing an element transformation matrix $\mathbf{T}^e$ that reduces the number of displacements

$$\bar{\mathbf{d}}^e = \mathbf{T}^e \mathbf{d}^e$$

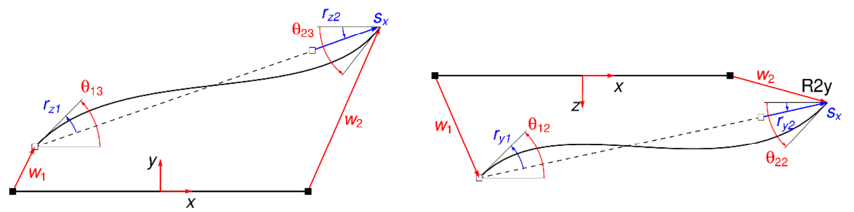
The transpose of the same matrix is used to reintroduce the rigid body modes when the element nodal forces need to be computed

$$\mathbf{f}^e = [\mathbf{T}^e]^T \bar{\mathbf{f}}^e$$

and when the element stiffness matrix is evaluated

$$\mathbf{K}^e = [\mathbf{T}^e]^T \bar{\mathbf{K}}^e \mathbf{T}^e$$

Elimination of rigid body modes



Element displacements are

$$\mathbf{d}^e = (w_{11},\, w_{12},\, w_{12},\, \theta_{11},\, \theta_{12},\, \theta_{13},\, w_{21},\, w_{22},\, w_{23},\, \theta_{21},\, \theta_{22},\, \theta_{23})$$

eliminating rigid body bodes one obtains a reduced set of displacement components

$$\bar{\mathbf{d}}^e = (s_x,\, r_x,\, r_{y1},\, r_{z1},\, r_{y2},\, r_{z2})$$

where

$$s_x = w_{21} - w_{11} \qquad r_{y1} = \theta_{12} - \frac{w_{13} - w_{23}}{L} \qquad r_{z1} = \theta_{13} + \frac{w_{12} - w_{22}}{L}$$

$$r_x = \theta_{21} - \theta_{11} \qquad r_{y1} = \theta_{y2} - \frac{w_{13} - w_{23}}{L} \qquad r_{z2} = \theta_{z2} + \frac{w_{12} - w_{22}}{L}$$

so that the transformation matrix $\mathbf{T}^e$ contains the coefficients of these relations.

Reintroducing rigid body modes

Rigid body modes need to be reintroduced when element forces are computed.

Actually, since the evaluation of the element response is made in a reference free of rigid body modes, only some components of element forces $\bar{\mathbf{f}}^e$ are computed,

$$\bar{\mathbf{f}}^e = (\bar{f}_x,\ \bar{m}_x,\ \bar{m}_{y1},\ \bar{m}_{z1},\ \bar{m}_{y2},\ \bar{m}_{z2})$$

which are dual to the displacements $\bar{\mathbf{d}}^e$.

Conversely, the solution procedure at the structural lever requires the determination of all components of the element forces

$$\mathbf{f}^e = (f_{x1},\ f_{y1},\ f_{z1},\ m_{x1},\ m_{y1},\ m_{z1},\ f_{x2},\ f_{y2},\ f_{z2},\ m_{x2},\ m_{y2},\ m_{z2})$$

In order to obtain $\mathbf{f}^e$ from $\bar{\mathbf{f}}^e$ duality between $\bar{\mathbf{d}}^e$ and $\bar{\mathbf{f}}^e$ and between $\mathbf{d}^e$ and $\mathbf{f}^e$ is employed

$$\bar{\mathbf{f}}^e \cdot \bar{\mathbf{d}}^e = \mathbf{f}^e \cdot \mathbf{d}^e \quad \Leftrightarrow \quad \bar{\mathbf{f}}^e \cdot (\mathbf{T}^e \mathbf{d}^e) = \mathbf{f}^e \cdot \mathbf{d}^e \quad \Leftrightarrow \quad ([\mathbf{T}^e]^T \bar{\mathbf{f}}^e) \cdot \mathbf{d}^e = \mathbf{f}^e \cdot \mathbf{d}^e$$

hence

$$\mathbf{f}^e = [\mathbf{T}^e]^T \bar{\mathbf{f}}^e$$

Reintroducing rigid body modes

The same transformation is used to obtain the element stiffness matrix.

Element equilibrium in the reference free of rigid body modes is expressed by

$$\bar{\mathbf{K}}_T^e \bar{\mathbf{d}}^e = \bar{\mathbf{f}}^e$$

where $\bar{\mathbf{d}}^e = \mathbf{T}^e \mathbf{d}^e$

$$\bar{\mathbf{K}}_T^e \mathbf{T}^e \mathbf{d}^e = \bar{\mathbf{f}}^e$$

Multiplying both terms by $[\mathbf{T}^e]^T$

$$[\mathbf{T}^e]^T \bar{\mathbf{K}}_T^e \mathbf{T}^e \mathbf{d}^e = [\mathbf{T}^e]^T \bar{\mathbf{f}}^e = \mathbf{f}^e$$

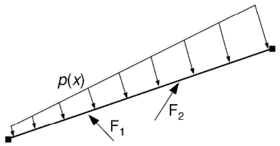
hence

$$\mathbf{K}_T^e = [\mathbf{T}^e]^T \bar{\mathbf{K}}_T^e \mathbf{T}^e$$

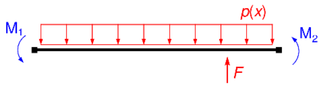
Effect of distributed loads along the element

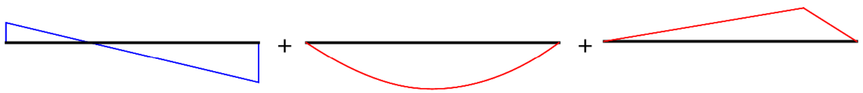
The distribution of internal forces can be determined by equilibrium and it is undependent from the material constitutive behaviour.

It is possible to account for distributed loads and forces applied at points internal to the element axis.



Interpolation of nodal forces and loads applied along the element





Internal forces are evaluated as:

$$\boldsymbol{\sigma}(x) = \mathbf{D}(x)\, \mathbf{f}^e + \mathbf{L}(x)\, \mathbf{l}$$

$\mathbf{D}$ = Interpolation function of nodal forces $\mathbf{f}^e$

$\mathbf{L}$ = Interpolation function od element loads $\mathbf{l}$

Variationally-based evaluation of the effects of element loads

The Principle of virtual forces is employed to define the element flexibility matrix and the element forces.

$$PVF: \int_0^L \delta\boldsymbol{\sigma} \cdot \Delta\mathbf{u}\, dx = \delta\bar{\mathbf{f}}^e \cdot \Delta\bar{\mathbf{d}}^e$$

where δ indicates the virtual quantities, which are evaluated in absence of element loads.
Hence $\delta\boldsymbol{\sigma}(x) = \mathbf{D}(x)\delta\bar{\mathbf{f}}^e$

$$\int_0^L \mathbf{D}\delta\bar{\mathbf{f}}^e \cdot \Delta\mathbf{u}\, dx = \delta\bar{\mathbf{f}}^e \cdot \Delta\bar{\mathbf{d}}^e \qquad \Rightarrow \qquad \int_0^L \mathbf{D}^T \Delta\mathbf{u}\, dx = \Delta\bar{\mathbf{d}}^e$$

Introducing the section constitutive law $\mathbf{K}^s \Delta\mathbf{u} = \Delta\boldsymbol{\sigma} \Leftrightarrow \Delta\mathbf{u} = [\mathbf{K}^s]^{-1}\Delta\boldsymbol{\sigma} = \mathbf{F}^s \Delta\boldsymbol{\sigma}$

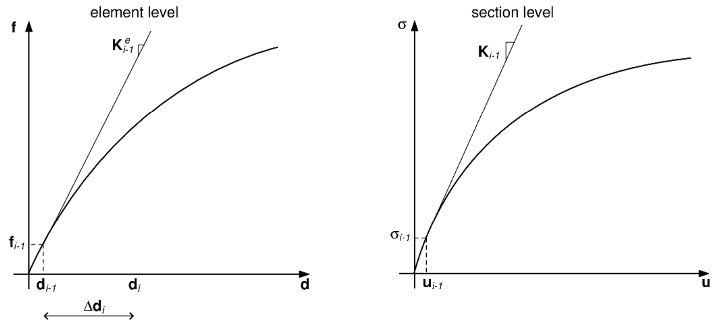
$$\int_0^L \mathbf{D}^T \mathbf{F}^s \Delta\boldsymbol{\sigma}\, dx = \Delta\bar{\mathbf{d}}^e \qquad \Rightarrow \qquad \int_0^L \mathbf{D}^T \mathbf{F}^s (\mathbf{D}\Delta\bar{\mathbf{f}}^e + \mathbf{L}\Delta\mathbf{l})\, dx = \Delta\bar{\mathbf{d}}^e$$

where we set $\boldsymbol{\sigma} = \mathbf{D}\bar{\mathbf{f}}^e + \mathbf{L}\mathbf{l}$.

Accordingly, the equilibrium of the element is written as

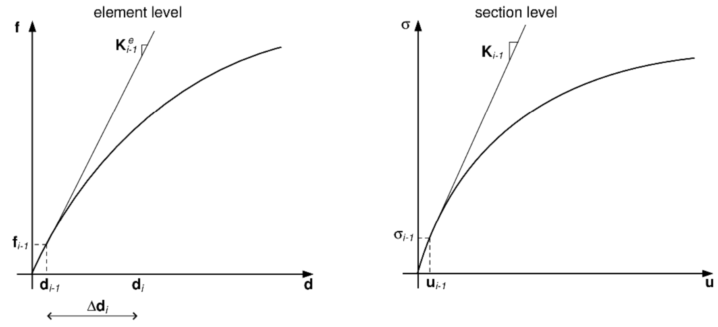
$$\bar{\mathbf{F}}^e \Delta\bar{\mathbf{f}}^e + \mathbf{Q}^e \Delta\mathbf{l} = \Delta\bar{\mathbf{d}}^e \quad \text{where} \quad \bar{\mathbf{F}}^e = \int_0^L \mathbf{D}^T \mathbf{F}^s \mathbf{D}\, dx \quad \mathbf{Q}^e = \int_0^L \mathbf{D}^T \mathbf{F}^s \mathbf{L}\, dx$$

Element state determination procedure



The element is used within a structural solution algorithm that adopts the tangent stiffness and the nodal forces of the structural model to solve the non-linear structural problem.

Element state determination procedure

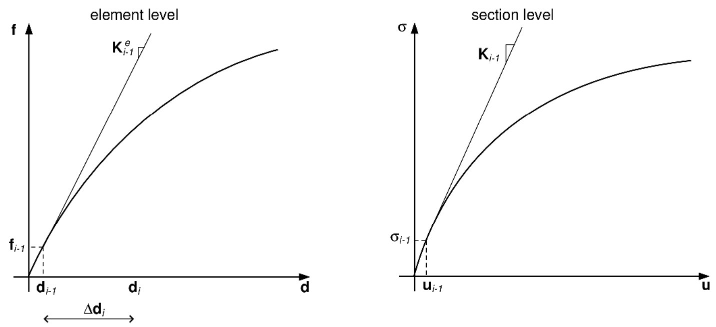


The element state determination procedure is invoked at the i-th iteration of the n-th step of the structural solution algorithm.

$\mathbf{d}_{i-1}$, $\mathbf{f}_{i-1}$, $\mathbf{K}^e_{i-1}$, $\mathbf{u}_{i-1}$, $\boldsymbol{\sigma}_{i-1}$ and $\mathbf{K}_{i-1}$ are known from the previous iteration.

New nodal displacements $\mathbf{d}_i$ are assigned to evaluate the element response in terms of $\mathbf{K}^e_i$ and $\mathbf{f}_i$.

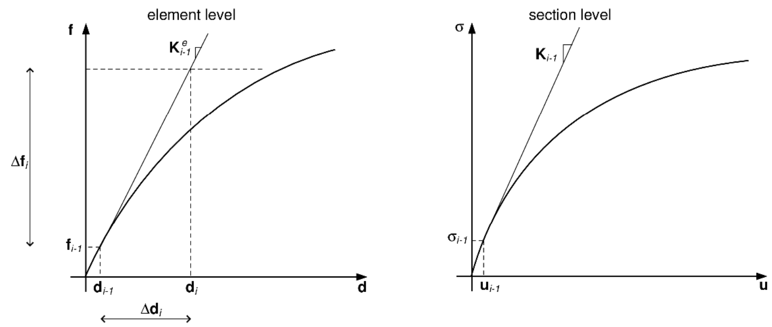
Element state determination procedure



Assigned nodal displacement $\mathbf{d}_i$ are used to evaluate the displacement increment $\Delta\mathbf{d}_i$ which are then transformed by eliminating the rigid body modes

$$\Delta\mathbf{d}_i = \mathbf{T}(\mathbf{d}_i - \mathbf{d}_{i-1})$$

Element state determination procedure

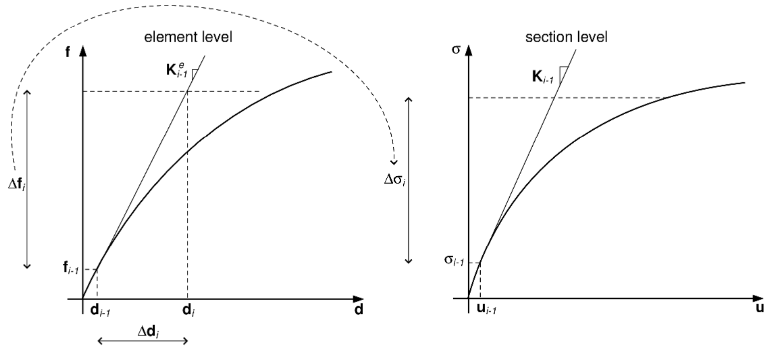


Adopting the element stiffness matrix at the previous iteration the force increment is evaluated as a function of $\Delta\mathbf{d}_i$

$$\Delta\mathbf{f}_i = \mathbf{K}^e_{i-1}\Delta\mathbf{d}_i$$

Remark: these force increments are not in equilibrium with the displacement increments $\Delta\mathbf{d}_i$ since they are evaluated with the stiffness matrix at the previous iteration.

Element state determination procedure

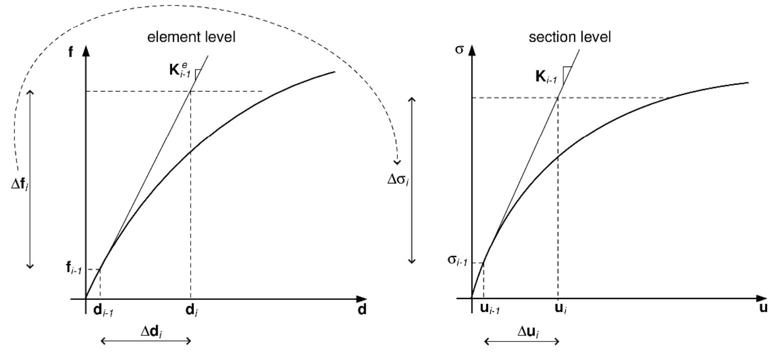


The internal forces increments at each control section of the element are evaluated by interpolation of element force increments

$$\Delta\boldsymbol{\sigma}_i = \mathbf{D}\Delta\mathbf{f}_i + \delta\boldsymbol{\sigma}_{i-1}$$

where $\delta\boldsymbol{\sigma}_{i-1}$ is a force corrector evaluated at the previous iteration.
Its evaluation at the current iteration is discussed later.

Element state determination procedure

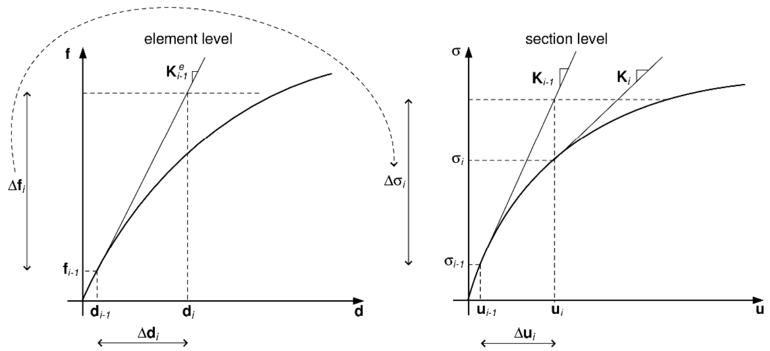


Section strain increments are evaluated by employing the section stiffness matrix at the previous iteration

$$\Delta\mathbf{u}_i = \mathbf{K}_{i-1}\Delta\boldsymbol{\sigma}_i \qquad \rightarrow \qquad \mathbf{u}_i = \mathbf{u}_{i-1} + \Delta\mathbf{u}_i$$

Remark: these strain increments are not in equilibrium with the force increments $\Delta\boldsymbol{\sigma}_i$ since they are evaluated with the stiffness matrix at the previous iteration.

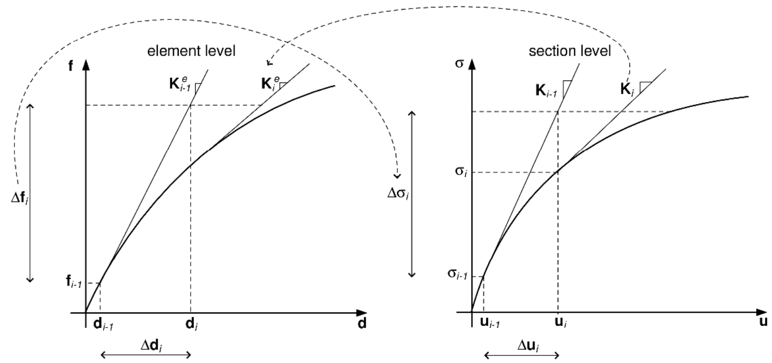
Element state determination procedure



These strain parameters are used to evaluate the internal forces and stiffness matrix of each control section

$$\boldsymbol{\sigma}_i = \boldsymbol{\sigma}(\mathbf{u}_i) \qquad \mathbf{K}_i = \mathbf{K}(\mathbf{u}_i)$$

Element state determination procedure

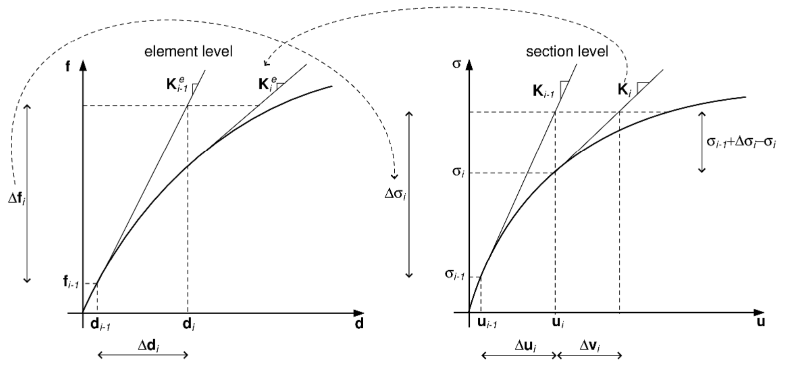


The section stiffness matrix is inverted and integrated along the element to evaluate the element flexibility matrix

$$\mathbf{F}^e = \int_0^L \mathbf{D}^T [\mathbf{K}_i^s]^{-1} \mathbf{D}\, dx$$

whose inverse is the element stiffness matrix $\mathbf{K}^e = [\mathbf{F}^e]^{-1}$.

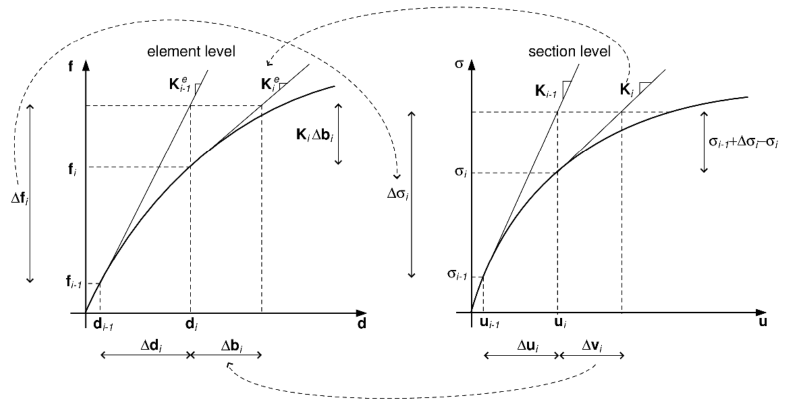
Element state determination procedure



If all the quantities above were evaluated considering the correct value of element and section stiffness (remarks above), the internal forces $\boldsymbol{\sigma}_i$ would correspond to $\boldsymbol{\sigma}_{i-1} + \Delta\boldsymbol{\sigma}_i$.
Conversely, a residual stress can be computed as $\boldsymbol{\sigma}_{i-1} + \Delta\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i$ and the relevant residual strain parameters are computed

$$\Delta\mathbf{v}_i = [\mathbf{K}_i]^{-1}(\boldsymbol{\sigma}_{i-1} + \Delta\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i)$$

Element state determination procedure



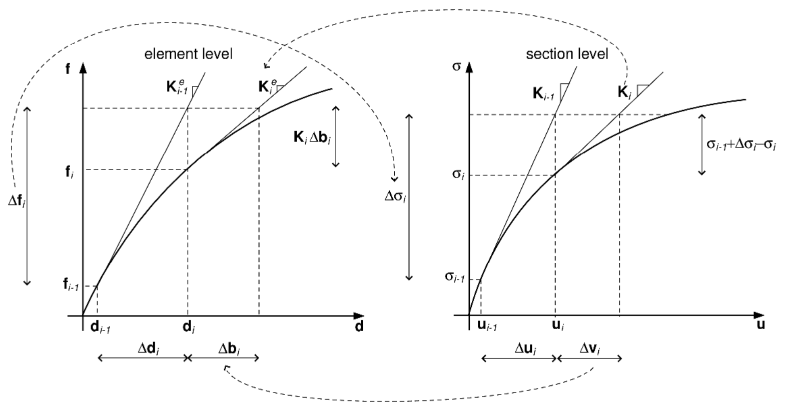
These residuals are integrated along the element to evaluate the displacement residuals

$$\mathbf{b}_i = \int_0^L \mathbf{D}^T \Delta\mathbf{v}_i \, dx$$

and transformed to residual forces $\mathbf{K}_i\Delta\mathbf{b}_i$ so as to evaluate the element forces

$$\mathbf{f}_i = \mathbf{f}_{i-1} + \Delta\mathbf{f}_i - \mathbf{K}_i\Delta\mathbf{b}_i$$

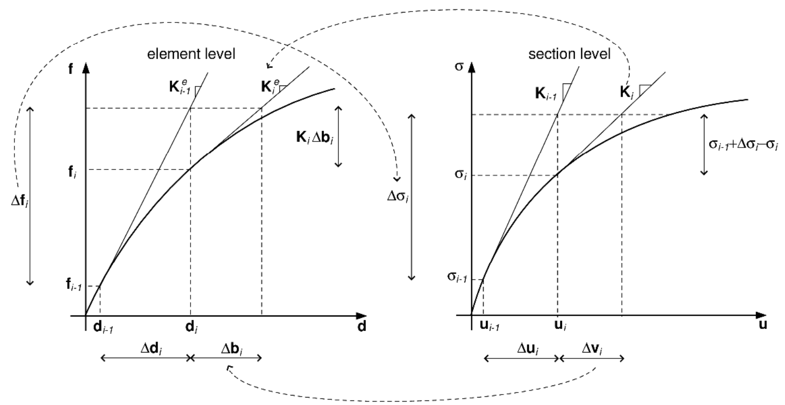
Element state determination procedure



An internal force corrector $\delta\boldsymbol{\sigma}_i$ is computed to be used within the next iteration. It is due to the fact that the integral forces $\boldsymbol{\sigma}_i$ do not correspond to an interpolation of the element forces $\mathbf{f}_i$, hence

$$\delta\boldsymbol{\sigma}_i = \mathbf{D}\mathbf{f}_i - \boldsymbol{\sigma}_i$$

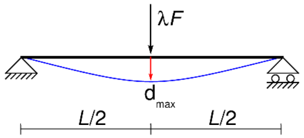
Element state determination procedure



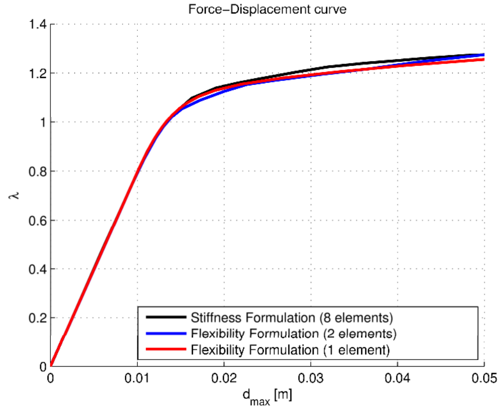
Finally, rigid body modes are reintroduced on both the element forces and stiffness matrix

$$\mathbf{f}^e_i = \mathbf{T}^T\mathbf{f}_i \qquad \mathbf{K}^e_i = \mathbf{T}^T\mathbf{K}_i\mathbf{T}$$

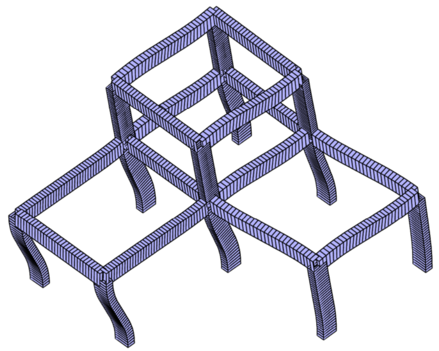
A simply supported beam

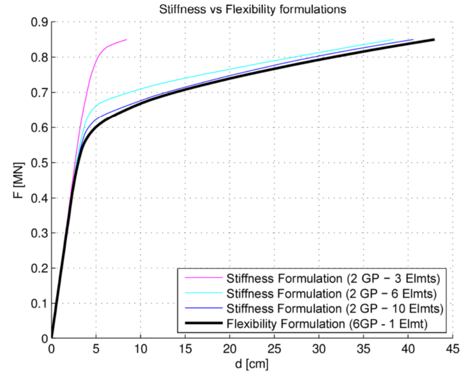


$L = 180$, $h = 10$, $b = 10$,
$E = 29000$, $\sigma_y = 50$,
$H_{iso} = 290$



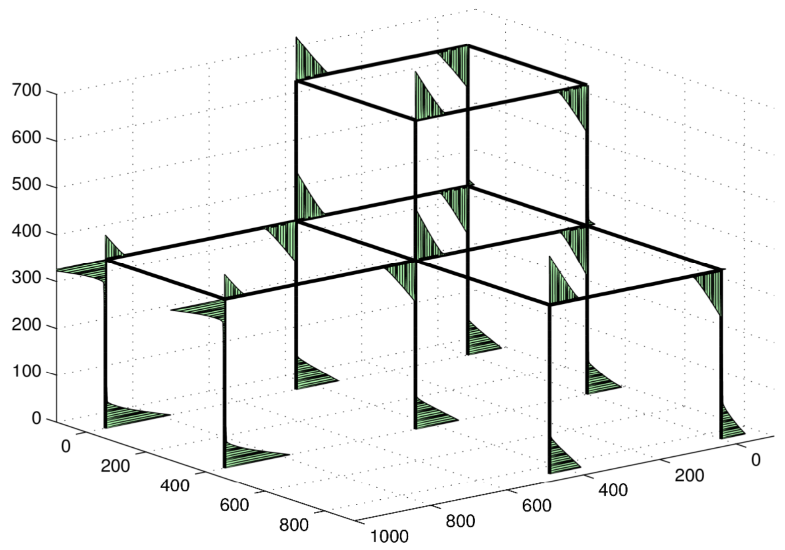
Analysis of a 3D RC frame



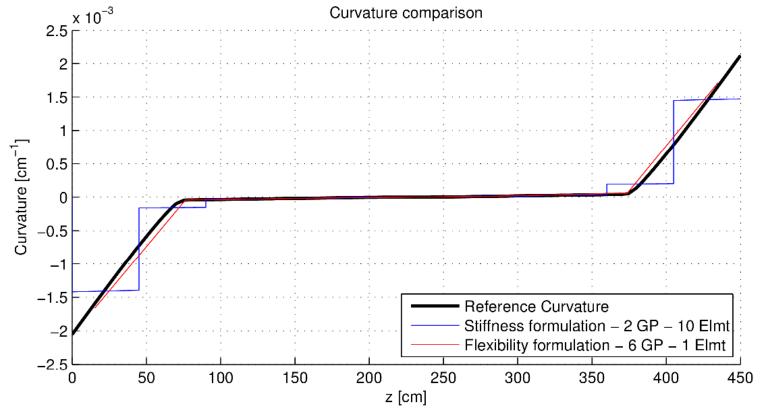


Stiffness formulations tend to overestimate structural stiffness and strength.
Flexibility formulations are more accurate even if only one elment per member is used.

Curvature diagram



Curvature diagram



Stiffness formulations furnish piecewise linear curvature distributions.
Flexibility formulations furnishes very accurate curvature distributions, even if only one element per member is used.

Quadrature rules at element level

Gauss (Gauss-Legendre) vs Lobatto (Gauss-Lobatto)

```
element dispBeamColumn $eleTag $iNode $jNode $numIntgrPts
                       $secTag $transfTag <-mass $massDens>
                       <-cMass> <-integration $intType>
```

```
element forceBeamColumn $eleTag $iNode $jNode $numIntgrPts
                        $secTag $transfTag <-mass $massDens>
                       <-iter $maxIters $tol> <-integration $intType>
```

How to set \$numIntgrPts and \$intType?

Evaluation of element integrals

The element state determination procedures of both stiffness and flexibility formulations require the evaluation of the following integrals

Stiffness formulation

$$\mathbf{q}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \boldsymbol{\sigma} dx$$

$$\mathbf{K}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \mathbf{K}_T \mathbf{B} dx$$

Flexibility formulation

$$\mathbf{b}^e = \int_{-L/2}^{L/2} \mathbf{D}^T \Delta \mathbf{v} dx$$

$$\mathbf{F}^e = \int_{-L/2}^{L/2} \mathbf{D}^T \mathbf{F} \mathbf{D} dx$$

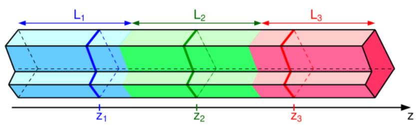
Quadrature rules for distributed plasticity elements

Element response (in terms of nodal forces **f**) and the relevant stiffness matrix **K** are expressed as 1D integrals defined along the element axis. Their evaluation requires numerical quadrature:

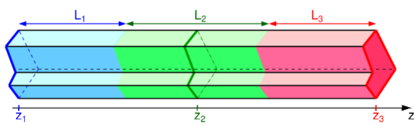
$$\int_{-1}^{1} p(\xi)\, d\xi = \sum_{i=1}^{n_q} p(\xi_i) w_i$$

Most common quadrature rules (for 1D elements) are:

Gauss: Abscissas z_i are all internal to the domain of integration and is exact for polynomial of degree $2n_q - 1$.



Lobatto: Two abscissas are forced to lie at element end points



Gauss abscissa and weights

n	ξ_i	w_i
1	0	2
2	$\pm 1/\sqrt{3}$	1
3	0	8/9
	$\pm\sqrt{3/5}$	5/9
4	$\pm\sqrt{\frac{3-2\sqrt{6/5}}{7}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3+2\sqrt{6/5}}{7}}$	$\frac{18-\sqrt{30}}{36}$

Lobatto abscissa and weights

n	ξ_i	w_i
1	–	–
2	± 1	1
3	± 1	1/3
	0	4/3
4	± 1	1/6
	$\pm\sqrt{5}/5$	5/6

Evaluation of element integrals

The element state determination procedures of both stiffness and flexibility formulations require the evaluation of the following integrals

Stiffness formulation

$$\mathbf{q}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \boldsymbol{\sigma} dx = \sum_{i=1}^{n_q} \mathbf{B}^T(\xi_i L/2)\boldsymbol{\sigma}(\xi_i L/2)\frac{Lw_i}{2}$$

$$\mathbf{K}^e = \int_{-L/2}^{L/2} \mathbf{B}^T \mathbf{K}_T \mathbf{B} dx = \sum_{i=1}^{n_q} \mathbf{B}^T(\xi_i L/2)\mathbf{K}_T(\xi_i L/2)\mathbf{B}(\xi_i L/2)\frac{Lw_i}{2}$$

Flexibility formulation

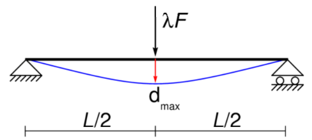
$$\mathbf{b}^e = \int_{-L/2}^{L/2} \mathbf{D}^T \Delta\mathbf{v} dx = \sum_{i=1}^{n_q} \mathbf{D}^T(\xi_i L/2)\Delta\mathbf{v}(\xi_i L/2)\frac{Lw_i}{2}$$

$$\mathbf{F}^e = \int_{-L/2}^{L/2} \mathbf{D}^T \mathbf{F} \mathbf{D} dx = \sum_{i=1}^{n_q} \mathbf{D}^T(\xi_i L/2)\mathbf{F}(\xi_i L/2)\mathbf{D}(\xi_i L/2)\frac{Lw_i}{2}$$

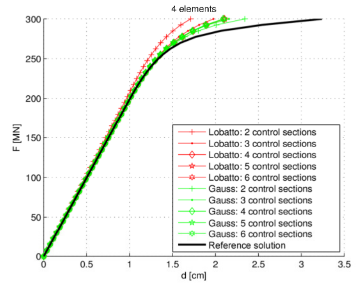
In both cases the quadrature rule is employed as

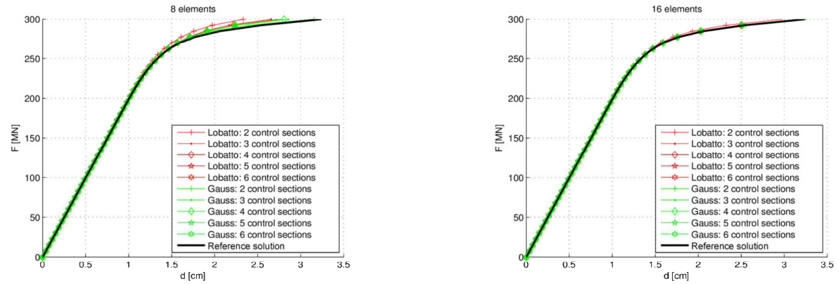
$$\int_{-1}^{1} p(\xi)\, d\xi = \sum_{i=1}^{n_q} p(\xi_i) w_i \quad \Rightarrow \quad \int_{-L/2}^{L/2} f(x) dx = \sum_{i=1}^{n_q} f(\xi_i L/2)\frac{Lw_i}{2}$$

Stiffness formulation: Gauss vs. Lobatto

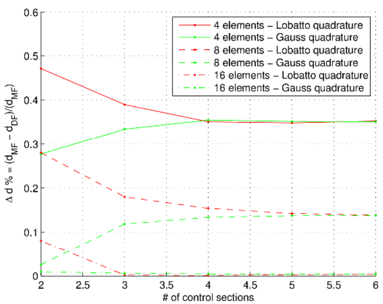


$L = 180,\ h = 10,\ b = 10,\ E = 29000,$
$\sigma_y = 50,\ H_{iso} = 290$





Stiffness formulation: Gauss vs. Lobatto



Usually internal forces attain higher values at the element extremities, hence:

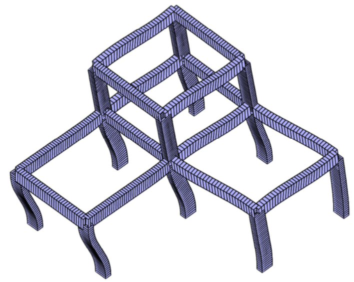
Lobatto

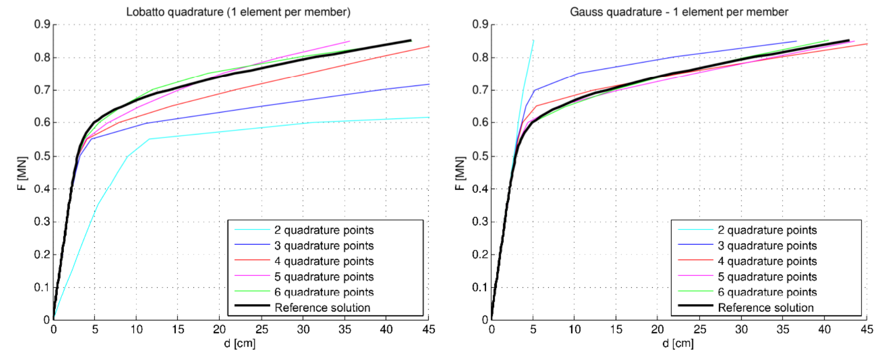
Two control section are placed at the element extremities; ⇒ higher values of section forces are sampled; ⇒ Reduction of strain values in order to re-establish the global equilibrium.

Gauss

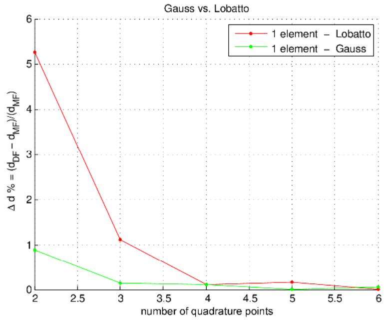
Control section are placed at the interior of the element; ⇒ lower values of section forces are sampled; ⇒ element forces ad stiffness are under-estimated.

Flexibility formulation: Gauss vs. Lobatto





Flexibility formulation: Gauss vs. Lobatto



Usually internal forces attain higher values at the element extremities, hence:

Lobatto
Two control section are placed at the element extremities; ⇒ higher values of section forces and element flexibility are computed; ⇒ element forces are increased to re-establish compatibility.

Gauss
Control section are placed at the interior of the element; ⇒ lower values of section forces and element flexibility are computed; ⇒ element forces are decreased to re-establish compatibility.

Evaluation of sectional response

The fiber method. Constitutive laws for steel and concrete

```
section  Fiber $secTag <-GJ $GJ> {
         fiber  ...
         patch  ...
         layer  ...
         ...
}

fiber $yLoc $zLoc $A $matTag

patch quad $matTag $numSubdivIJ $numSubdivJK
           $yI $zI $yJ $zJ $yK $zK $yL $zL

patch rect $matTag $numSubdivY $numSubdivZ $yI $zI $yJ $zJ

patch circ $matTag $numSubdivCirc $numSubdivRad
           $yCenter $zCenter $intRad $extRad $startAng $endAng

layer straight $matTag $numFiber $areaFiber
               $yStart $zStart $yEnd $zEnd

layer circ $matTag $numFiber $areaFiber
           $yCenter $zCenter $radius <$startAng $endAng>
```


Evaluation of the sectional response

Both the stiffness and flexibility formulations require the evaluation of the sectional response

Stiffness formulation

$$\mathbf{K}^e = \int_L \mathbf{B}^T \mathbf{K}^s \mathbf{B} dz\,, \quad \mathbf{f}^e = \int_L \mathbf{B}^T \boldsymbol{\sigma} dz$$

Flexibility formulation

$$\mathbf{F}^e = \int_L \mathbf{D}^T [\mathbf{K}^s]^{-1} \mathbf{D} dz\,, \quad \mathbf{Q} = \int_L \mathbf{D}^T [\mathbf{K}^s]^{-1} \mathbf{L}\, dz$$

$$\boldsymbol{\sigma} = \begin{matrix} N \\ -M_y \\ M_x \end{matrix} \begin{pmatrix} \int_A \sigma[\varepsilon(y,z)]\, dA \\ \int_A \sigma[\varepsilon(y,z)] x\, dA \\ \int_A \sigma[\varepsilon(y,z)] y\, dA \end{pmatrix}$$

$$\boldsymbol{\sigma}(x) = \mathbf{D}(x)\,\mathbf{f} + \mathbf{L}(x)\,\mathbf{l}$$

$$\mathbf{K}^s_{nl} = \begin{pmatrix} \int_A E_t[\varepsilon(x,y)]\, dA & \int_A E_t[\varepsilon(y,z)] y\, dA & \int_A E_t[\varepsilon(y,z)] z\, dA \\ \int_A E_t[\varepsilon(y,z)] y\, dA & \int_A E_t[\varepsilon(y,z)] y^2\, dA & \int_A E_t[\varepsilon(y,z)] yz\, dA \\ \int_A -E_t[\varepsilon(y,z)] y\, dA & \int_A E_t[\varepsilon(y,z)] yz\, dA & \int_A E_t[\varepsilon(y,z)] z^2\, dA \end{pmatrix}$$

Evaluation of the sectional response

Both the stiffness and flexibility formulations require the evaluation of the sectional response

Stiffness formulation

$$\mathbf{r} = (y,\, z)^T \qquad \boldsymbol{\rho} = (1,\, y,\, z)^T$$

$$\mathbf{K}^e = \int_L \mathbf{B}^T \mathbf{K}^s \mathbf{B} dz\,, \quad \mathbf{f}^e = \int_L \mathbf{B}^T \boldsymbol{\sigma} dz$$

Flexibility formulation

$$\boldsymbol{\sigma} = \int_A \sigma[\varepsilon(\mathbf{r})] \boldsymbol{\rho}\, dA$$

$$\mathbf{F}^e = \int_L \mathbf{D}^T [\mathbf{K}^s]^{-1} \mathbf{D} dz\,, \quad \mathbf{Q} = \int_L \mathbf{D}^T [\mathbf{K}^s]^{-1} \mathbf{L}\, dz$$

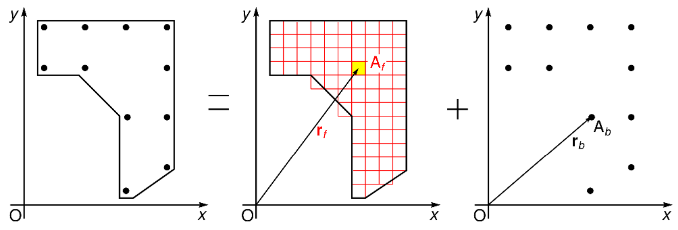
$$\boldsymbol{\sigma}(x) = \mathbf{D}(x)\,\mathbf{f} + \mathbf{L}(x)\,\mathbf{l}$$

$$\mathbf{K}^s_{nl} = \int_A E_t[\varepsilon(\mathbf{r})] \boldsymbol{\rho} \otimes \boldsymbol{\rho}\, dA$$

Evaluation of the sectional response

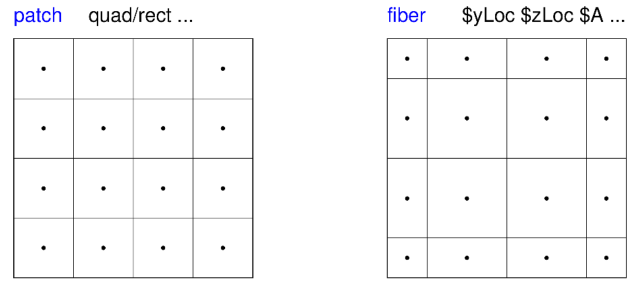
$$\boldsymbol{\sigma}^s_{nl} = \sum_{f=1}^{n_f} A_f \sigma[\varepsilon(\mathbf{r}_f)]\boldsymbol{\rho}_f + \sum_{b=1}^{n_b} A_b \sigma[\varepsilon(\mathbf{r}_b)]\boldsymbol{\rho}_b$$

$$\mathbf{K}^s_{nl} = \sum_{f=1}^{n_f} A_f E_t[\varepsilon(\mathbf{r}_f)]\boldsymbol{\rho}_f \otimes \boldsymbol{\rho}_f + \sum_{b=1}^{n_b} A_b E_t[\varepsilon(\mathbf{r}_b)]\boldsymbol{\rho}_b \otimes \boldsymbol{\rho}_b$$



Position of fibers

Automatic positioning vs. manual positioning of fibers at Gauss points?



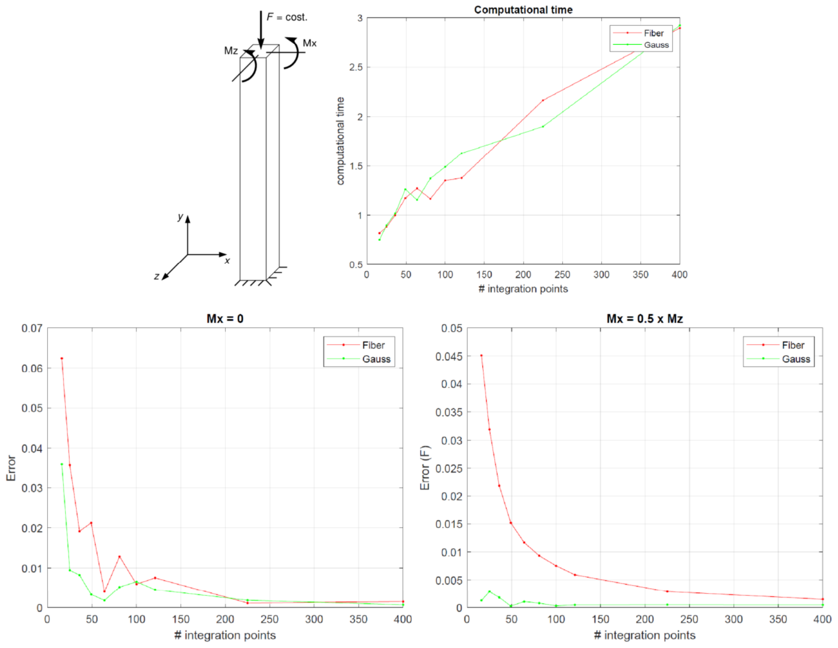

```
section Fiber $secTag <-GJ $GJ> {
    patch quad $matTag $numSubdivIJ $numSubdivJK
               $yI $zI $yJ $zJ $yK $zK $yL $zL
}
```

```
section Fiber $secTag <-GJ $GJ> {
    patch rect $matTag $numSubdivY $numSubdivZ $yI $zI $yJ $zJ
}
```

```
section Fiber $secTag <-GJ $GJ> {
    fiber $yLoc $zLoc $A $matTag
    fiber $yLoc $zLoc $A $matTag
    ...
}
```


Position of fibers

Comparison by D. Pellecchia (UniNA)

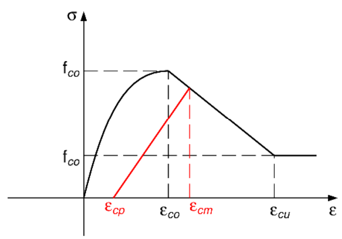


Materials for RC

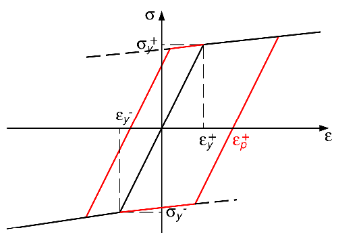
$$\boldsymbol{\sigma}^s_{nl} = \sum_{f=1}^{n_f} A_f \sigma[\varepsilon(\mathbf{r}_f)]\boldsymbol{\rho}_f + \sum_{b=1}^{n_b} A_b \sigma[\varepsilon(\mathbf{r}_b)]\boldsymbol{\rho}_b$$

$$\mathbf{K}^s_{nl} = \sum_{f=1}^{n_f} A_f E_t[\varepsilon(\mathbf{r}_f)]\boldsymbol{\rho}_f \otimes \boldsymbol{\rho}_f + \sum_{b=1}^{n_b} A_b E_t[\varepsilon(\mathbf{r}_b)]\boldsymbol{\rho}_b \otimes \boldsymbol{\rho}_b$$

Concrete (uniaxial materials)



Steel reinforcements (uniaxial materials)




```
uniaxialMaterial Concrete01 $matTag $fpc $epsc0 $fpcu $epsU
```

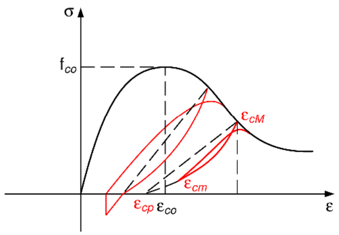
```
uniaxialMaterial Steel01 $matTag $Fy $E0 $b <$a1 $a2 $a3 $a4>
```

Materials for RC

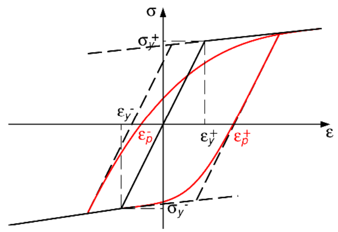
$$\boldsymbol{\sigma}^s_{nl} = \sum_{f=1}^{n_f} A_f \sigma[\varepsilon(\mathbf{r}_f)]\boldsymbol{\rho}_f + \sum_{b=1}^{n_b} A_b \sigma[\varepsilon(\mathbf{r}_b)]\boldsymbol{\rho}_b$$

$$\mathbf{K}^s_{nl} = \sum_{f=1}^{n_f} A_f E_t[\varepsilon(\mathbf{r}_f)]\boldsymbol{\rho}_f \otimes \boldsymbol{\rho}_f + \sum_{b=1}^{n_b} A_b E_t[\varepsilon(\mathbf{r}_b)]\boldsymbol{\rho}_b \otimes \boldsymbol{\rho}_b$$

Concrete (uniaxial materials)



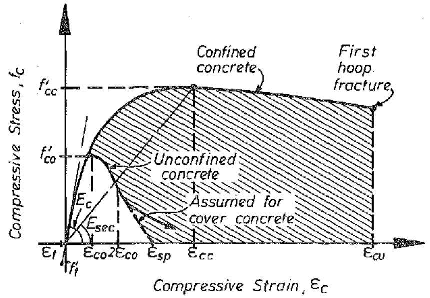
Steel reinforcements (uniaxial materials)

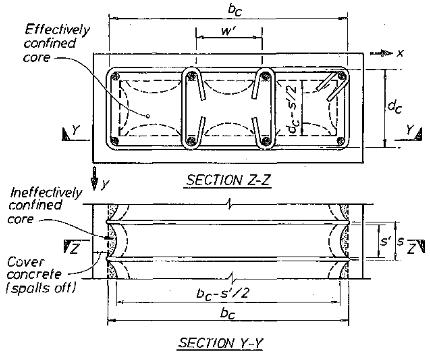



```
uniaxialMaterial Concrete04 $matTag $fc $ec $ecu $Ec
                            <$fct $et> <$beta>
```

```
uniaxialMaterial Steel02 $matTag $Fy $E $b $R0 $cR1 $cR2
                         <$a1 $a2 $a3 $a4 $sigInit>
```


Effect of confinement: Mander's Law



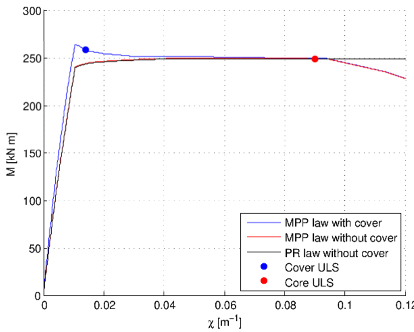


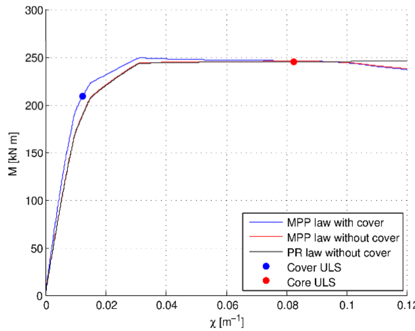
$$\sigma_c = \frac{f'_{cc}\, r \frac{\varepsilon}{\varepsilon_{cc}}}{r - 1 + \left(\frac{\varepsilon}{\varepsilon_{cc}}\right)^r} \qquad \varepsilon_{cc} = \varepsilon_{co}\left[1 + 5\left(\frac{f'_{cc}}{f'_{co}} - 1\right)\right]$$

$$r = \frac{E_c}{E_c - E_{sec}} \qquad E_{sec} = \frac{f'_{cc}}{\varepsilon_{cc}} \qquad E_c = 5000\sqrt{f'_{c0}}\ [MPa] \qquad \varepsilon_{cu} = 0.004 + 0.084\frac{\rho_s f_{yh}}{f'_{cc}}$$

$$f'_{cc} = f'_{co}\left(-1.254 + 2.254\sqrt{1 + 7.94\frac{f'_l}{f'_{co}}} - 2\frac{f'_l}{f'_{co}}\right) \qquad f'_l = k_e \rho_s f_{yh}/2$$

Effect of confinement: Mander's Law

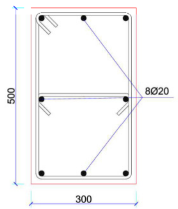


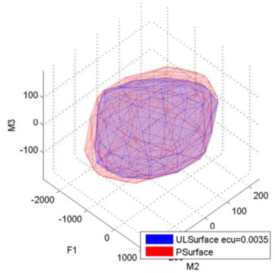


Section size 300 $mm \times$ 500 mm;
Concrete strength $f_{cd} = 8.3\ MPa$;
Reinforcement 8∅20 mm bars (4 at corners + 4 at side midpoints);
Reinforcement strength $f_{yd} = 450\ MPa$;
Transversal reinforcement: ∅8 mm / 150 mm (rectangular stirrup + cross tie).

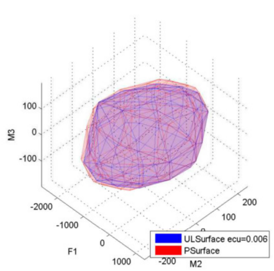
Concrete peak stress $f'_{cc} = 11.947\ MPa$;
Concrete ultimate strength $f'_{cu} = 11.817\ MPa$;
Concrete ultimate strain $\varepsilon_{cu} = 0.0198$.

Effect of confinement: Comparison with Plastic domains

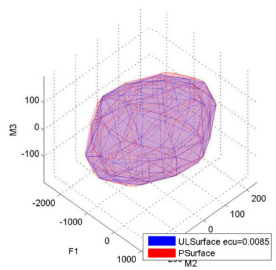




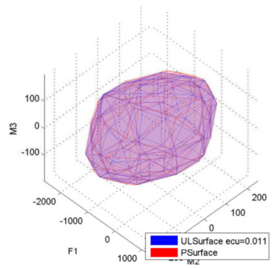
(a) $\varepsilon_{cu} = 0.35\%$



(b) $\varepsilon_{cu} = 0.60\%$

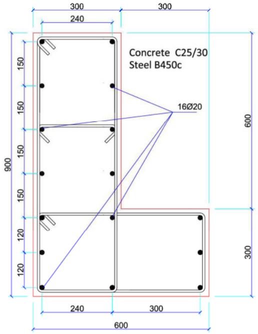


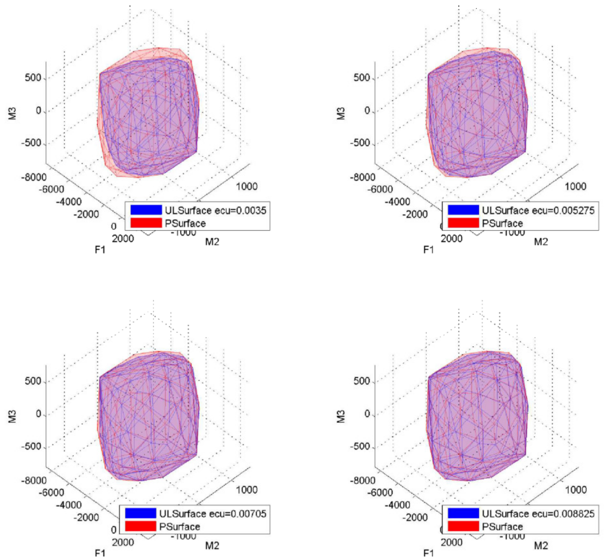
(c) $\varepsilon_{cu} = 0.85\%$



(d) $\varepsilon_{cu} = 1.10\%$

Effect of confinement: Comparison with Plastic domains





Dynamic analysis

The central difference method and the Newmark's method

```
integrator CentralDifference

integrator Newmark $gamma $beta
```


Numerical evaluation of dynamic response

The equation of motion (dynamic equilibrium equation) of the structure reads

$$\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C}\dot{\mathbf{d}}(t) + \mathbf{f}[\mathbf{d}(t)] = -\mathbf{M}\ddot{\mathbf{d}}_g(t) = \mathbf{f}_g(t)$$

Time stepping methods consist in discretizing the time t in intervals $\Delta t = t_{i+1} - t_i$.

Numerical procedures for solving the equation of motion are required to *converge* to the exact solution as Δt is decreased, be *accurate* and *stable* in presence of numerical roud-off.

Stability is a critical aspect since these procedures are often *conditionally stable* due the requirement $\Delta t < \Delta t_{\text{crit.}}$

Two families of methods are considered:

Explicit methods All quantities at time t_{i+1} are evaluated as a function of quantities at time t_i.

Implicit methods Quantities at time t_{i+1} are evaluated iteratively as a function of both quantities at times t_i and t_{i+1}.

The central difference method (explicit)

Time derivative are approximated by finite differences

$$\dot{\mathbf{d}}_{i+1} = \frac{\mathbf{d}_{i+1} - \mathbf{d}_{i-1}}{2\Delta t} \qquad \ddot{\mathbf{d}}_{i+1} = \frac{1}{\Delta t}\left(\frac{\mathbf{d}_{i+1} - \mathbf{d}_i}{\Delta t} - \frac{\mathbf{d}_i - \mathbf{d}_{i-1}}{\Delta t}\right) = \frac{\mathbf{d}_{i+1} - 2\mathbf{d}_i + \mathbf{d}_{i-1}}{\Delta t^2}$$

Substitution into the equation of motion yields

$$\mathbf{M}\frac{\mathbf{d}_{i+1} - 2\mathbf{d}_i + \mathbf{d}_{i-1}}{\Delta t^2} + \mathbf{C}\frac{\mathbf{d}_{i+1} - \mathbf{d}_{i-1}}{2\Delta t} + \mathbf{f}[\mathbf{d}_i] = \mathbf{f}_{gi}$$

which is rewritten by factoring $\mathbf{d}_{i+1}$ as

$$\left(\frac{\mathbf{M}}{\Delta t^2} + \frac{\mathbf{C}}{2\Delta t}\right)\mathbf{d}_{i+1} + \mathbf{f}(\mathbf{d}_i) - \mathbf{f}_{gi} - 2\frac{\mathbf{M}}{\Delta t^2}\mathbf{d}_i + \frac{\mathbf{M}}{\Delta t^2}\mathbf{d}_{i-1} - \frac{\mathbf{C}}{2\Delta t}\mathbf{d}_{i-1} = \mathbf{0}$$

Setting

$$\hat{\mathbf{f}}_i = \mathbf{f}_{gi} - \mathbf{f}(\mathbf{d}_i) + 2\frac{\mathbf{M}}{\Delta t^2}\mathbf{d}_i - \frac{\mathbf{M}}{\Delta t^2}\mathbf{d}_{i-1} + \frac{\mathbf{C}}{2\Delta t}\mathbf{d}_{i-1} \qquad \hat{\mathbf{K}} = \frac{\mathbf{M}}{\Delta t^2} + \frac{\mathbf{C}}{2\Delta t}$$

the equation of motion becomes

$$\hat{\mathbf{K}}\mathbf{d}_{i+1} = \hat{\mathbf{f}}_i \qquad \rightarrow \qquad \mathbf{d}_{i+1} = \hat{\mathbf{K}}^{-1}\hat{\mathbf{f}}_i$$

The central difference method (explicit): initial conditions

At first step the procedure requires the quantities $\mathbf{d}_o$ and $\dot{\mathbf{d}}_o$, which are usually given, and $\ddot{\mathbf{d}}_o$, $\mathbf{d}_{-1}$ which are evaluated as follows:

The approximate finite differences

$$\dot{\mathbf{d}}_o = \frac{\mathbf{d}_1 - \mathbf{d}_{-1}}{2\Delta t} \qquad \ddot{\mathbf{d}}_o = \frac{\mathbf{d}_1 - 2\mathbf{d}_o + \mathbf{d}_{-1}}{\Delta t^2}$$

are used to evaluate

$$\mathbf{d}_{-1} = \mathbf{d}_o - \dot{\mathbf{d}}_o \Delta t + \frac{\Delta t^2}{2}\ddot{\mathbf{d}}_o$$

The equation of motion at time 0 is solved to have

$$\mathbf{M}\ddot{\mathbf{d}}_o + \mathbf{C}\dot{\mathbf{d}}_o + \mathbf{f}(\mathbf{d}_o) = \mathbf{f}_{go} \qquad \rightarrow \qquad \ddot{\mathbf{d}}_o = \mathbf{M}^{-1}[\mathbf{f}_{go} - \mathbf{C}\dot{\mathbf{d}}_o - \mathbf{f}(\mathbf{d}_o)]$$

The Newmark's method (implicit)

At each time step Δt, the equation of motion is solved by an iterative procedure that is similar to the Newton method and assuming

$$\dot{\mathbf{d}}_{i+1} = \dot{\mathbf{d}}_i + [(1-\gamma)\ddot{\mathbf{d}}_i + \gamma\ddot{\mathbf{d}}_{i+1}]\Delta t \qquad \text{and} \qquad \mathbf{d}_{i+1} = \mathbf{d}_i + \dot{\mathbf{d}}_i\Delta t + \frac{1}{2}[(1-\beta)\ddot{\mathbf{d}}_i + \beta\ddot{\mathbf{d}}_{i+1}]\Delta t^2$$

in which it is typically assumed $\gamma = 1/2$ and $\beta = 1/3$, ..., $1/2$.

To this end the equation of motion is written at time t_{i+1} in the form of a residual

$$\mathbf{D}(\mathbf{d}_{i+1}) = \mathbf{M}\ddot{\mathbf{d}}_{i+1} + \mathbf{C}\dot{\mathbf{d}}_{i+1} + \mathbf{f}(\mathbf{d}_{i+1}) - \mathbf{f}_{gi+1} = \mathbf{0}$$

The Taylor series expansion of the residual $\mathbf{D}(\mathbf{d}_{i+1})$ yields

$$\mathbf{D}(\mathbf{d}_{i+1}) \approx \mathbf{D}(\mathbf{d}^o_{i+1}) + \frac{\partial \mathbf{D}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} \delta\mathbf{d}_{i+1} = \mathbf{D}(\mathbf{d}^o_{i+1}) + \left[\mathbf{M}\frac{\partial \ddot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} + \mathbf{C}\frac{\partial \dot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} + \frac{\partial \mathbf{f}(\mathbf{d}_{i+1})}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}}\right]\delta\mathbf{d}_{i+1}$$

Recalling the expression of $\mathbf{d}_{i+1}$ we have

$$\ddot{\mathbf{d}}_{i+1} = \frac{2}{\beta\Delta t^2}(\mathbf{d}_{i+1} - \mathbf{d}_i) - \frac{2}{\beta\Delta t}\dot{\mathbf{d}}_i - \left(\frac{1}{\beta} - 1\right)\ddot{\mathbf{d}}_i \qquad \Rightarrow \qquad \frac{\partial \ddot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} = \frac{2}{\beta\Delta t^2}$$

Substituting this expression of $\ddot{\mathbf{d}}_{i+1}$ into $\dot{\mathbf{d}}_{i+1}$ yields

$$\dot{\mathbf{d}}_{i+1} = \frac{2\gamma}{\beta\Delta t}(\mathbf{d}_{i+1} - \mathbf{d}_i) + \left(1 - \frac{2\gamma}{\beta}\right)\dot{\mathbf{d}}_i + \left(1 - \frac{\gamma}{\beta}\right)\ddot{\mathbf{d}}_i\Delta t \qquad \Rightarrow \qquad \frac{\partial \dot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} = \frac{2\gamma}{\beta\Delta t}$$

The Newmark's method (implicit)

Thus, considering

$$\frac{\partial \ddot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} = \frac{2}{\beta\Delta t^2}, \qquad \frac{\partial \dot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} = \frac{2\gamma}{\beta\Delta t}, \qquad \frac{\partial \mathbf{f}(\mathbf{d}_{i+1})}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} = \mathbf{K}^o_{i+1}$$

the Taylor series expansion of the residual $\mathbf{D}(\mathbf{d}_{i+1})$ becomes

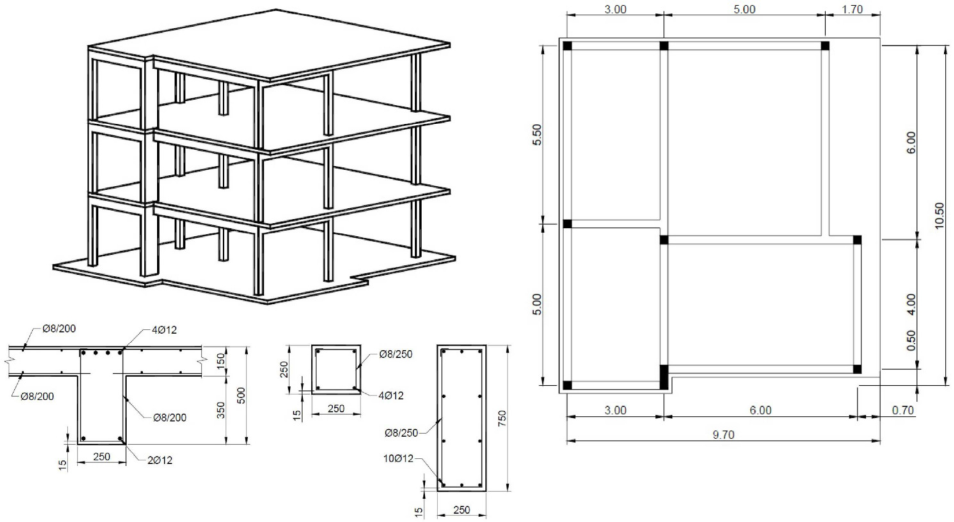
$$\begin{aligned}\mathbf{D}(\mathbf{d}_{i+1}) &\approx \mathbf{D}(\mathbf{d}^o_{i+1}) + \left[\mathbf{M}\frac{\partial \ddot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} + \mathbf{C}\frac{\partial \dot{\mathbf{d}}_{i+1}}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}} + \frac{\partial \mathbf{f}(\mathbf{d}_{i+1})}{\partial \mathbf{d}_{i+1}}\bigg|_{\mathbf{d}^o_{i+1}}\right]\delta\mathbf{d}_{i+1} \\ &= \mathbf{D}(\mathbf{d}^o_{i+1}) + \left[\mathbf{M}\frac{2}{\beta\Delta t^2} + \mathbf{C}\frac{2\gamma}{\beta\Delta t} + \mathbf{K}^o_{i+1}\right]\delta\mathbf{d}_{i+1} \\ &= \mathbf{D}(\mathbf{d}^o_{i+1}) + \hat{\mathbf{K}}^o_{i+1}\delta\mathbf{d}_{i+1}\end{aligned}$$

Setting $\mathbf{D}(\mathbf{d}_{i+1}) = \mathbf{0}$ the iterative displacement increment is computed as

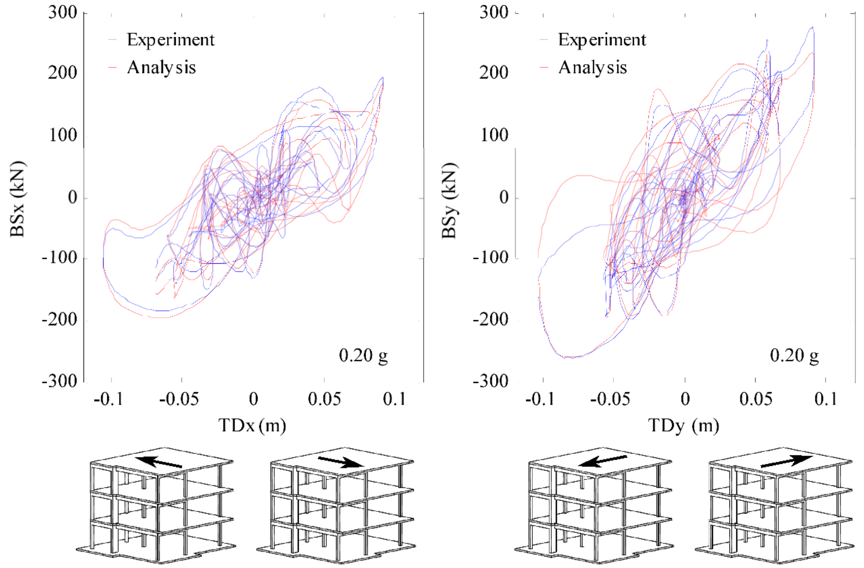
$$\delta\mathbf{d}_{i+1} = -[\hat{\mathbf{K}}^o_{i+1}]^{-1}\mathbf{D}(\mathbf{d}^o_{i+1}) \qquad \rightarrow \qquad \delta\mathbf{d}^j_{i+1} = -[\hat{\mathbf{K}}^{j-1}_{i+1}]^{-1}\mathbf{D}(\mathbf{d}^{j-1}_{i+1})$$

where $\mathbf{D}(\mathbf{d}^{j-1}_{i+1})$ is obtained by substituting the expression of $\ddot{\mathbf{d}}^{j-1}_{i+1}$ and $\dot{\mathbf{d}}^{j-1}_{i+1}$ given at the end of the previous slide into the expression of $\mathbf{D}(\mathbf{d}_{i+1})$.

Pushover analysis of a 3D RC frame: The SPEAR building



An example: The SPEAR building (Dolšek and Fajfar)



Bibliography

Finite element analysis of structures

- http://opensees.berkeley.edu/wiki/
- E.L. Wilson, 2002. Three-dimensional static and dynamic analysis of structures. Computers and Structures, Inc.
- A.K. Chopra, 2011. Dynamics of structures. Prentice-Hall
- T.J.R. Hughes, 1987. The finite element method. Linear Static and dynamic finite element analysis. Prentice-Hall.
- M.A. Crisfield, 2000. Non-linear Finite Element Analysis of Solids and Structures. Volumes 1 and 2. John Wiley and sons.
- O.C. Zienkiewics, R.L. Taylor, 2000. The finite element method. Volume 1: The basis. Butterworth Heinemann.

Bibliography

Flexibility formulations

- E. Spacone, V. Ciampi, F.C. Filippou, 1996. Mixed formulation of nonlinear beam finite element. Computers & Structures, 58, 71-83.
- A. Neuenhofer, F.C. Filippou, 1997. Evaluation of nonlinear frame finite-element models. ASCE Journal of Structural Engineering, 123, 958-966.
- A. Neuenhofer, F.C. Filippou, 1998. Geometrically nonlinear flexibility-based frame finite element. ASCE Journal of Structural Engineering, 124, 704-711.
- J. Coleman, E. Spacone, 2001. Localization issues in force-based frame elements. ASCE Journal of Structural Engineering, 127, 1257-1265.
- F. Marmo, L. Rosati, 2012. An improved fexibility-based nonlinear frame element endowed with the fiber-free formulation. 6th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS), Vienna, Austria, September 10th-14th.

Bibliography

Evaluation of sectional response

- P.J. Davis, P. Rabinowitz, 2007. Methods of Numerical Integration, (2nd edn). Dover Publications.
- M. Saje, I. Turk, B. Vratanar, 1997. A kinematically exact finite element formulation of planar elastic–plastic frames. Computer Methods in Applied Mechanics and Engineering, 144:125–151.
- F. Marmo, L. Rosati, 2012. Analytical integration of elasto-plastic uniaxial constitutive laws over arbitrary sections. International Journal for Numerical Methods in Engineering, 91:990-1022.
- F. Marmo, L. Rosati, 2013. The fiber-free approach in the evaluation of the tangent stiffness matrix for elasto-plastic uniaxial constitutive laws. International Journal for Numerical Methods in Engineering, 94:868-894.
- S. Sessa, F. Marmo, L. Rosati, L. Leonetti, G. Garcea, R. Casciaro, 2018. Evaluation of the capacity surfaces of reinforces concrete sections: Eurocode versus a plasticity-based approach. Meccanica, 53:1493-1512.